AFCRC-TN-56-799
ASTIA DOCUMENT No. AD110140

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-100

Asymptotic Expansion of Multiple Integrals and the Method of Stationary Phase

DOUGLAS S. JONES and MORRIS KLINE

CONTRACT No. AF 19(604)1717
OCTOBER 1956

NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

Research Report No. EM-100

ASYMPTOTIC EXPANSION OF MULTIPLE INTEGRALS AND THE METHOD OF STATIONARY PHASE

Douglas S. Jones and Morris Kline

Douglas S. Jones

Morris Kline
Project Director

October, 1956

The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF19(604)1717. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1956

Table of Contents

Abstract

This paper considers the asymptotic evaluation of the double integral

$$\iint_D g(x,y)e^{ikf(x,y)}dydx$$

for large k. The domain D is finite and the functions g and f are assumed to be analytic in and on the boundary of D. Under these conditions certain points (x,y) of D, called critical points, prove to be decisive. The asymptotic form of the double integral is given by the sum of asymptotic series in integral and fractional powers of 1/k, and each of these series is determined by a neighborhood in D of a critical point. The method of this paper is the essentially new feature. The problem of evaluating the double integral asymptotically is reduced to the problem of evaluating a single Fourier integral asymptotically and known results for the latter case are applied.

1. Introduction

In problems of the diffraction theory of optics* as well as in many micro-wave problems a solution is obtained in the form

$$(1) \qquad J = \iint_D g(x,y)e^{ikf(x,y)}dydx,$$

wherein g and f are real functions and generally k is real. Physically, $g(x,y)$ is an amplitude function, $f(x,y)$ is a phase function and $k = 2\pi/\lambda$, λ being the wavelength of some source which gives rise to the field represented by the double integral. It is usually impossible to evaluate integrals of this form explicitly. In the applications mentioned above one is interested in the value of the integral for small wavelength, that is, large k**. Hence in recent years attention has been focused on the problem of obtaining an asymptotic series representation of J in integral and fractional powers of $1/k$.

To obtain such a result Van Kampen[4] applied the method of stationary phase, originally developed for single integrals, in a purely formal manner. It is clear from the formal uses of the method of stationary phase that the asymptotic series representation of (1) is a sum of asymptotic series determined by the behavior of $f(x,y)$ in the neighborhood of certain critical points of the domain D of integration.

Recently Focke[6] gave a rigorous treatment of the asymptotic expansion of J in which he includes a number of types of critical points not previously treated. He uses the notion of a neutralizer originally introduced by van der Corput for single integrals and extended by Focke to double integrals. The neutralizer is a function of x and y which serves the purpose of isolating the various critical points so that one might determine the contribution each makes to the asymptotic

* See, for example, the review articles by E. Wolf[1] and H. Bremmer[2] and also a recent thesis by J. Berghuis[3].

**The case of small k is also of practical interest in optics. See the work of K. Nienhuis and B.R.A. Nijboer, referred to by Wolf[1].

evaluation of the double integral.

Both Focke and the present paper assume that the functions g and f of the integral are analytic in and on the boundary of the region D. Braun[10], using the method of van der Corput, assumes only that f and g have a finite number of continuous derivatives; he obtains explicit results only for some types of interior critical points, and indicates the method to be employed for some boundary critical points. The method and the results are complicated but may be the best obtainable on the basis of his weaker assumptions. A number of special results on the asymptotic evaluation of the integral J obtained by Berghuis[3], Bremmer[5], Siegel[12], and Kontorovitch and Muravév[14] are germane. Mention should also be made of a forthcoming report by N. Chako who is pursuing other methods of evaluating the integral J asymptotically.

The present paper is designed to show that the asymptotic expansion of the integral J can be reduced almost at once to the asymptotic expansion of single Fourier integrals, that is, integrals of the form

$$(2) \qquad \int_{\alpha}^{\beta} h(t)\, e^{ikt} dt,$$

wherein h(t) and k are real. The asymptotic expansion of such integrals and of the more general integrals

$$(3) \qquad \int_{\alpha}^{\beta} h(t)\, e^{ikg(t)} dt$$

has been extensively investigated. However, using the notion of a neutralizer and integration by parts, Erdélyi[7] has given a direct proof of the key theorem on asymptotic expansion of integrals of the forms (2) and (3). The theorem of Erdélyi, which deals with integrals of the form (2) and on which this paper rests, reads as follows:

Theorem. If $\phi(t)$ is N times continuously differentiable for $\alpha \leq t \leq \beta$, and $0 < \lambda \leq 1$, $0 < \mu \leq 1$, then

$$\int_{\alpha}^{\beta} e^{ikt}(t-\alpha)^{\lambda-1}(\beta-t)^{\mu-1}\phi(t)dt = B_N(k) - A_N(k) + O(k^{-N}) \quad \text{as } k \to \infty,$$

where

$$A_N(k) = \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!} e^{i\pi(n+\lambda-2)/2} k^{-n-\lambda} e^{ik\alpha} \frac{d^n}{d\alpha^n}\left[(\beta-\alpha)^{\mu-1}\phi(\alpha)\right]$$

$$B_N(k) = \sum_{n=0}^{N-1} \frac{\Gamma(n+\mu)}{n!} e^{i\pi(n-\mu)/2} k^{-n-\mu} e^{ik\beta} \frac{d^n}{d\beta^n}\left[(\beta-\alpha)^{\lambda-1}\phi(\beta)\right],$$

and $O(k^{-N})$ may be replaced by $o(k^{-N})$ if $\lambda = \mu = 1$.

If the original integral (2) is to be considered over a larger domain $a \leq \alpha < \beta \leq b$, then, of course, it is to be decomposed into subdomains in each of which the above theorem applies. Also if $\phi(t)$ is infinitely differentiable within $\alpha < t < \beta$ then $A_N(k)$ and $B_N(k)$ become infinite series and the order of the remainder becomes less than any power of $1/k$.

The method of this paper is certainly no more complicated than Focke's. Moreover it seems to have several advantages. The analysis shows how the behavior of the contour lines of $f(x,y)$, that is, the lines $f(x,y)$ = const., determines the critical points. These lines have immediate physical significance. In diffraction optics $f(x,y)$ represents an optical distance from the source to a given point in the image space along a ray whose first two direction cosines in the image space are x and y. It is therefore possible with the present theory to predict the types of critical points from a knowledge of the rays and to interpret the contribution of each critical point in terms of the behavior of rays. Secondly, the calculation of the successive coefficients of the various asymptotic series arising from the several critical points is simpler in the present case. Thirdly, the reduction of the problem of asymptotic expansion of higher dimensional multiple integrals of the form (1) to that of single integrals is immediately effected by the method of this paper.

2. Reduction of the problem to the single integral case

We consider integrals of the form

$$J = \iint_D g(x,y)e^{ikf(x,y)}\,dxdy,$$

where g and f are real analytic functions, thus possessing derivatives of all orders, at all points interior to and on the boundary of D. The boundary of D is assumed to be piecewise infinitely differentiable, and such that if $\phi(x,y) = 0$ is a representation of any piece, ϕ_x and ϕ_y are both zero at any point on the boundary. We shall take k to be real, though the simpler case of complex k is actually included, as noted below.

Let m and M be the smallest and largest values attained by f(x,y) in D so that $m \leq f(x,y) \leq M$ for x, y in D. Then, for $\varepsilon > 0$,*

$$e^{ikf} = \int_{m-\varepsilon}^{M+\varepsilon} e^{ikt}\delta(t-f)dt. \tag{4}$$

Hence

$$J = \iint_D g\, e^{ikf}dxdy = \iint_D g \int_{m-\varepsilon}^{M+\varepsilon} e^{ikt}\delta(t-f)dt\, dxdy. \tag{5}$$

By an interchange of the order of integration we obtain

$$J = \int_{m-\varepsilon}^{M+\varepsilon} e^{ikt} \iint_D g\, \delta(t-f)dxdydt, \tag{6}$$

---

*We write $m-\varepsilon$ and $M+\varepsilon$ merely to emphasize that the behavior of f at the end-points m and M is significant.

or

(7) $$J = \int_{m-\varepsilon}^{M+\varepsilon} e^{ikt} h(t)\,dt,$$

where

(8) $$h(t) = \iint_D g(x,y)\, \delta \left\{ t - f(x,y) \right\} dxdy.$$

Thus the problem of obtaining the asymptotic behavior of J is reduced to that of evaluating the single Fourier integral on the right side of equation (7). The same reduction is obviously possible in the case of integrals of several variables.

It follows from Erdélyi's theorem that there are contributions to the asymptotic expansion of an integral of the form (7) only from those critical values of t, possibly including the end points m and M, at which h(t) or any derivative is discontinuous or at which h(t) or any derivative becomes infinite. In the case of infinities the theorem gives the form of the expansion only for integrable algebraic singularities of h(t). The case of a logarithmic infinity in h(t) will also be needed and will be treated separately in the Appendix*. Since for the h(t) of this paper, namely (8) above, $h(m-\varepsilon) = h(M+\varepsilon) = 0$, we need consider just those values of t within the closed interval (m,M).

It is intuitively clear - the precise analysis will be given later - that the behavior of h(t) at any value t_o of t will depend upon the behavior of the contour lines $f(x,y) = t$ for t near and at t_o. Moreover, on the contour line $f(x,y) = t_o$ only certain points and their neighborhoods will be significant. For example, if at an interior point (x_o,y_o) of D at least one of the partial derivatives f_x and f_y is not zero, then the contour lines through (x_o,y_o) and points in a small neighborhood will be smooth and change smoothly with t, so that h(t) and its successive derivatives will be continuous insofar as contributions to h(t) from this neighborhood

*Independently, A. Erdélyi has extended his theorem to the case of logarithmic singularities of h(t) in (2) and (3). His results and the results of our Appendix overlap. His results appear in the Journal of the Society for Industrial and Applied Mathematics, March, 1956.

of (x_o,y_o) are concerned. On the other hand, if (x_o,y_o) is a relative maximum point of $f(x,y)$, so that f_x and f_y are zero there, then $h(t)$ will be zero for $t > t_o$ insofar as contributions to $h(t)$ from a neighborhood of (x_o,y_o) are concerned, and $h(t)$ may well be analytic for $t < t_o$. In any case $h(t)$ will not be analytic at $t = t_o$. In other words, the non-analyticity of $h(t)$ at any value t_o of t will depend upon the behavior of the contour lines $f(x,y) = t$ only in the neighborhoods of certain points (x_o,y_o), called critical points, a fact already observed in earlier formal treatments of the asymptotic expansion of J. Similar remarks apply to integrals of more than two variables when the curves f = const. are replaced by surfaces or hypersurfaces f = const.

When k is complex and $0 < \arg k < \pi$, it is necessary to consider the behavior of $h(t)$ only near $t = m$ because the contributions from other points are exponentially small compared with the expansion obtained from $t = m$. Thus we evaluate the integral expression (8) for $h(t)$ near $t = m$ and, on account of the presence of the δ-function, we need use only a neighborhood of $f = m$. The behavior of the contour lines $f(x,y)$ = const. in this neighborhood will determine the asymptotic expansion. When $-\pi < \arg k < 0$, we need consider only the behavior of $h(t)$ near $t = M$.

Since, as is indicated by the preceding intuitive evidence, the significant behavior of $h(t)$ will depend upon its behavior in the neighborhoods of certain critical points of D we shall develop some new forms for $h(t)$ which will be useful in discussing its behavior in small regions of D. Let D_o be any (small) subdomain of D and let us introduce a coordinate transformation, whose nature will be specified later, from x, y to X, Y. Let $f(x,y)$ transform to $F(X,Y)$ and let the product of $g(x,y)$ and the Jacobian of the transformation transform to $G(X,Y)$. Then, letting $h_o(t)$ represent the value of $h(t)$ over D_o, we have

$$h_o(t) = \iint_{D_o} G(X,Y)\, \delta\left\{ t - F(X,Y) \right\} dX\, dY.$$

We now set

$$F(X,Y) = F_o(X,Y) + F_1(X,Y)$$

and introduce the new variables

(9) $$\xi = F_o(X,Y), \qquad \eta = \Upsilon(X,Y),$$

where Υ is at our disposal. Let $\mathcal{F}_1(\xi,\eta)$ and $\mathcal{G}(\xi,\eta)$ be the functions which result from F_1 and G under this change of variables. Then

(10) $$h_o(t) = \iint_{D_o} \mathcal{G}(\xi,\eta) \frac{\partial(X,Y)}{\partial(\xi,\eta)} \delta\left\{ t-\xi - \mathcal{F}_1(\xi,\eta)\right\} d\xi d\eta ,$$

where D_o now represents the (ξ,η) domain corresponding to the original (X,Y) domain.

We apply Taylor's theorem to the δ-function in (10) and expand around $t = \xi$ with $-\mathcal{F}_1$ as the increment. Then

(11) $$h_o(t) = \sum_{r=0}^{\infty} \iint_{D_o} \frac{(-1)^r}{r!} \mathcal{G}(\xi,\eta) \frac{\partial(X,Y)}{\partial(\xi,\eta)} \mathcal{F}_1^r(\xi,\eta)\delta^{(r)}(t-\xi)d\xi d\eta ,$$

where $\delta^{(r)}$ indicates the r-th derivative of $\delta(t-\xi)$.

It may be helpful towards understanding later steps if before transforming further we examine the meaning of the equation just obtained. The right side is a sum of terms of the form

(12) $$\iint_{D_o} \mathcal{H}(\xi,\eta)\ \delta^{(r)}(t-\xi)d\xi d\eta \quad .$$

The double integral may be regarded as a repeated integral with respect to ξ and then with respect to η. Hence the inner integral is of the form

(13) $$\int_{\xi_1(c)}^{\xi_2(c)} \mathcal{H}(\xi,c)\ \delta^{(r)}(t-\xi)d\xi,$$

where $\xi_1(c)$ and $\xi_2(c)$ are the least and greatest values of ξ for $\eta = c$ in the domain D_o. By utilizing a property of the δ-function [9] we may convert this integral to some function $\mathcal{K}(t,\eta)$ and we must then consider

$$\int_{\eta_1}^{\eta_2} \mathcal{K}(t,\eta)d\eta ,$$

where η_1 and η_2 are the least and greatest values of η in all the (ξ,η) for which $\xi = t$. For arbitrary t, η_1 and η_2 will be functions of t.

If we proceed to transform equation (11) in the manner indicated, we shall encounter some difficulty*. Instead we shall reverse the order of integration in (12). Thus

---

* The difficulty we seek to avoid is the following. Let us consider the term in (11) when r = 1. We should now use the relevant property of the δ-function

$$\int_a^b f(x)\delta'(t-x)dx = f'(t) + f(a)\delta(t-a) - f(b)\delta(t-b) \quad .$$

This is obtained by integration by parts from the equation defining the δ-function, namely,

$$\int_a^b f(x)\delta\ t-x)dx = f(t).$$

When we apply the above property to the case r = 1 in (13) we obtain

$$\mathcal{H}_t(t,\eta) + \mathcal{H}\Big(\xi_1(c),\eta\Big) \quad \delta\Big(t-\xi_1(c)\Big) - \mathcal{H}\Big(\xi_2(c),\eta\Big) \quad \delta\Big(t-\xi_2(c)\Big) \quad .$$

If we now integrate with respect to η , the second and third terms may contribute to the result because c must now be replaced by η . For r > 1, the difficulties in this procedure become considerably greater.

$$\iint_{D_o} \mathcal{H}(\xi,\eta)\delta^{(r)}(t-\xi)\,d\xi\,d\eta = \int_{\xi_1}^{\xi_2} \delta^{(r)}(t-\xi) \int_{\eta_1(\xi)}^{\eta_2(\xi)} \mathcal{H}(\xi,\eta)\,d\eta\,d\xi;$$

$$= \int_{\xi_1}^{\xi_2} \delta^{(r)}(t-\xi)\,\mathcal{K}(\xi)\,d\xi;$$

$$= \frac{\partial^r}{\partial t^r}\,\mathcal{K}(t)^* ;$$

$$= \frac{\partial^r}{\partial t^r} \int_{\eta_1(t)}^{\eta_2(t)} \mathcal{H}(t,\eta)\,d\eta .$$

If we apply this result to each term of (11) we obtain the result

$$(14) \qquad h_o(t) = \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \int_{\eta_1(t)}^{\eta_2(t)} \frac{(-1)^r}{r!}\,\mathcal{H}(t,\eta)\,\frac{\partial(X,Y)}{\partial(t,\eta)}\,\mathcal{F}_1^r(t,\eta)\,d\eta .$$

Formula (14) for $h_o(t)$ is basic in the subsequent calculations. Its merit as opposed to that of formula (8) is that the path of integration is now $t = \xi$ instead of $t = f(x,y)$. However it will be convenient to introduce one or two variations of it. By reason of the analyticity of f and g we can say that in the neighborhood of any point (X_o,Y_o), which corresponds to (x_o,y_o), the following absolutely convergent expansions hold:

---

* This result is valid for t between ξ_1 and ξ_2. There will be a few cases in which we shall want the behavior of $h_o(t)$ at $t = \xi_1$ or $t = \xi_2$. However, $h_o(t)$ will be singular at such values of t. But the behavior at $t = \xi_1+$ or $t = \xi_2-$, which is what will actually be needed later, is determined by the behavior of $h_o(t)$ for $t > \xi_1+$ or $t < \xi_2-$. Hence we need not consider the value of $h_o(t)$ for $t = \xi_1$ or $t = \xi_2$.

$$G(X,Y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G_{m,n}(X-X_o)^m(Y-Y_o)^n$$

and

$$\frac{(-1)^r}{r!} F_1^r(X,Y) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} F_{r,p,q}(X-X_o)^p(Y-Y_o)^q .$$

Therefore

(15) $$\frac{(-1)^r}{r!} G(X,Y)F_1^r(X,Y) = \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} (X-X_o)^\lambda(Y-Y_o)^\mu \sum_{p=0}^{\lambda} \sum_{q=0}^{\mu} G_{\lambda-p,\mu-q} F_{r,p,q} .$$

If we now introduce the variables ξ, η of (9) in place of X, Y and then write t for ξ we shall have the integrand required in (14), apart from the Jacobian.

If, in particular, it should be the case in (9) that

(16) $$X - X_o = \xi_1(\xi)\ \eta_1(\eta) \quad \text{and} \quad Y - Y_o = \xi_1(\xi)\ \eta_2(\eta)$$

then (15) becomes

(17) $$\frac{(-1)^r}{r!} \mathcal{B}(\xi,\eta)\ \mathcal{F}_1^r(\xi,\eta) = \sum_{\mu=0}^{\infty} \xi_1^\mu \sum_{\lambda=0}^{\mu} \eta_1^\lambda\ \eta_2^{\mu-\lambda} \sum_{p=0}^{\lambda} \sum_{q=0}^{\mu-\lambda} G_{\lambda-p,\mu-\lambda-q} F_{r,p,q} ,$$

and, if we replace ξ by t, we again have the form of the integrand required in formula (14), except for the Jacobian.

3. The behavior of h(t) at ordinary points of D

We propose now to characterize the critical points (x,y) of D, to show that if f(x,y) = t contains the critical point then h(t) is analytic, and to prove that if (x,y) is a critical point then the resulting non-analytic form of h(t) is determined by an arbitrarily small neighborhood of (x,y).

Let us label as ordinary or non-critical, those interior points (x,y) of D at which at least one of f_x and f_y is not zero and those boundary points at which the same condition holds, the boundary is analytic, and f(x,y) = const. is neither tangent to nor coincident with the boundary. All other points or arcs (in the case of coincidence of part of $f(x,y) = $ const. with the boundary) will be called critical points or arcs. We shall assume that the number of these critical points or arcs is finite.

Now let t_o be any value of t such that $f(x,y) = t_o$ does not contain any critical points or arcs. Then there will exist a small neighborhood, $|t-t_o| < \delta$, such that for any t in this neighborhood $f(x,y) = t$ does not contain any critical points or arcs. Let

$$\xi = f(x,y) \qquad \text{and} \qquad \eta = \Psi(x,y),$$

where $\Psi(x,y) = $ const. is the family of orthogonal trajectories to the family of curves $f(x,y) = t$. Since $f(x,y)$ is analytic in and on the boundary of D, $f(x,y)$ is analytic in a small neighborhood of each boundary point of D. Hence the orthogonal trajectories $\Psi(x,y)$ are defined even for (x,y) outside of but sufficiently near D. The transformation from (x,y) to (ξ,η) is analytic, one-to-one, and its inverse is also analytic.

Let us call D_1 the subdomain of D containing all (x,y) for which $f(x,y) = t$, where $|t-t_o| < \delta$. We may therefore write, in view of (8),

$$h_1(t) = \iint_{D_1} G(\xi,\eta)\delta(t-\xi)\,\frac{\partial(x,y)}{\partial(\xi,\eta)}\,d\xi d\eta$$

or

$$h_1(t) = \int_{\eta_1(t)}^{\eta_2(t)} G(t,\eta)\,\frac{\partial(x,y)}{\partial(t,\eta)}\,d\eta\,. \tag{18}$$

If the contour line $f(x,y) = t_o$ is a closed curve (see Fig. 1a) then the (ξ,η)-subdomain corresponding to the subdomain $f(x,y) = t$ with $|t-t_o| < \delta$ will

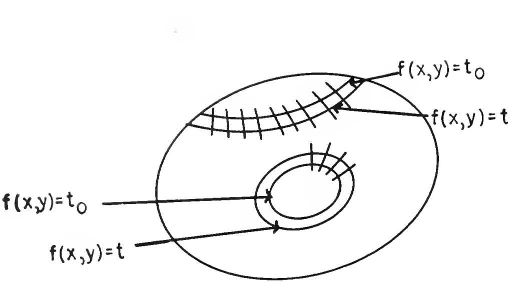

Fig. 1a

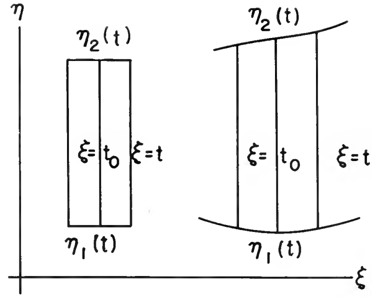

Fig. 1b

be a rectangle in the (ξ,η)-plane with boundaries $\eta_1(t)$ and $\eta_2(t)$ which are horizontal line segments* (see Fig. 1b). If the contour line $f(x,y) = t_o$ is an arc which cuts the boundary of D, that is, it is not tangent there, then the subdomain $f(x,y) = t$ with $|t-t_o| < \delta$ will be a region in the (ξ,η)-plane bounded on the left and right by vertical line segments, and above and below by arcs $\eta_1(t)$ and $\eta_2(t)$, which are analytic since the boundary of D is analytic at the points of intersection. In either case $h(t)$ is an analytic function of t. If the contour line $f(x,y) = t_o$ should consist of several disjoint curves the same argument applies to each.

---

* The family of orthogonal trajectories will exist and yield the necessary analyticity and one-to-one correspondence between (x,y) and (ξ,η) only in an (x,y) domain where $f_x \neq 0$. Hence it may be necessary to break up the arc or closed curve $f(x,y) = t_o$ into a sum of arcs with some arcs containing the points (x,y) at which $f_x(x,y) = 0$ as interior points. Then the domain D_1 will be come a sum of domains and the integral (8) will be a sum of integrals. To treat the transformation from (x,y) to (ξ,η) for a subdomain containing an arc of $f(x,y) = t_o$ on which $f_x = 0$, we introduce a rotation of coordinates which makes both f_x and $f_y \neq 0$. The integral (18) becomes a sum of integrals each of which is analytic in t.

Now let t_o be a value of t such that there is a critical point (x_o,y_o) on the contour line $f(x,y) = t_o$ (see Fig. 2). Let us again consider a small interval of t-values, $|t-t_o| < \delta$. We shall isolate this critical point (x_o,y_o) by one or more arcs such that for any value of t in this interval $f(x,y) = t$ will be decomposed into two parts, one lying within a neighborhood of (x_o,y_o) bounded by these arcs and another lying outside.

We require that these arcs be analytic*. Then h(t) will consist of two integrals, one of the form (18) which will be analytic and another, $h_o(t)$, given by some integral of the form (8) but extending only over those (x,y)-values lying within the region bounded by the arcs isolating (x_o,y_o).

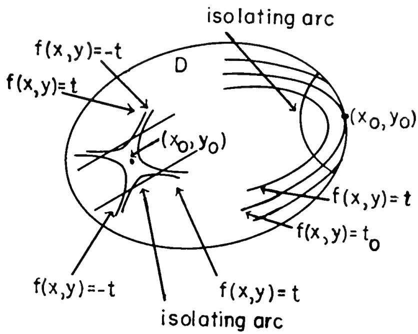

Fig. 2

We shall see later that $h_o(t)$ is non-analytic for $t = t_o$. Moreover, it follows that the non-analytic form of $h_o(t)$ is independent of the choice of the arcs isolating (x_o,y_o) provided only that they be analytic.

We have therefore shown that the values of t at which h(t) is not analytic are limited to those values of t for which $f(x,y) = t$ contains one or more critical points (x,y). Further, if t_o is indeed a value of t at which h(t) is not analytic, then the non-analytic part of h(t) in the neighborhood of $t = t_o$ is determined by an arbitrarily small neighborhood in D of the critical point (x,y) which lies on $f(x,y) = t_o$.

* The precise choice of the shapes of these arcs will be specified in considering the various types of critical points. We need to be careful later only to observe the analyticity condition to be imposed on these arcs. The decomposition of the path $f(x,y) = t$ into two parts will be needed to treat only some of the critical points to be considered later.

4. Contributions from critical, non-stationary boundary points

Let us use the term non-stationary point for points (x,y) at which at least one of the partial derivatives f_x and f_y is not zero. We have just seen that a non-stationary point (x_o,y_o) is ordinary if it is an interior point of D or if it is a boundary point for which the contour line $f(x,y) = \text{const.}$ through it intersects the boundary and if the boundary is analytic at the intersection. There may, however, by non-stationary boundary points which are critical.

4.1 We consider a non-stationary boundary point (x_o,y_o) such that the boundary is analytic at (x_o,y_o) and such that the contour line $f(x,y) = t_o$ through (x_o,y_o) is tangent to the boundary. Let us suppose that $f_x(x_o,y_o) \neq 0$ and that the equation of the boundary curve is $\phi(x,y) = 0$. We shall first show how to choose new coordinates X, Y such that for the transform $F(X,Y)$ of $f(x,y)$, F_{10} or $\partial F/\partial X$ at $(X_o,Y_o) \neq 0$, $F_{01}(X_o,Y_o) = 0$, the boundary $\phi(x,y) = 0$ becomes the $X = 0$ axis, and the positive direction of the X-axis points into D.

Since the contour line $f(x,y) = t_o$ is tangent to the boundary at (x_o,y_o) we have at (x_o,y_o)

$$\begin{vmatrix} f_x & f_y \\ \phi_x & \phi_y \end{vmatrix} = 0.$$

We make the linear transformation*

$$\begin{aligned} x - x_o &= au - m\phi_y v \\ y - y_o &= cu + m\phi_x v \end{aligned} \quad , \qquad \begin{vmatrix} a & -m\phi_y \\ c & m\phi_x \end{vmatrix} = 1 ,$$

with the choice of a and c such as to make the positive u-axis point into D.

Since

$$f(x,y) = f_{00} + f_{10}(x-x_o) + f_{01}(y-y_o) + f_{20}(x-x_o)^2 + f_{11}(x-x_o)(y-y_o) + \dots ,$$

---

* This transformation is suggested by Focke[6], p. 38.

then, if $\bar{f}(u,v)$ denotes the transform of $f(x,y)$,

(19) $$\bar{f}(u,v) = \bar{f}_{00} + \bar{f}_{10}u + \bar{f}_{20}u^2 + \bar{f}_{11}uv + \cdots \ .$$

Moreover

$$\bar{\phi}(u,v) = \bar{\phi}_{10}u + \bar{\phi}_{20}u^2 + \bar{\phi}_{11}uv + \cdots \ .$$

To make the boundary curve of D the X = 0 axis, we introduce X, Y through

(20) $$\bar{\phi}_{10}X = \bar{\phi}_{10}u + \bar{\phi}_{20}u^2 + \bar{\phi}_{11}uv + \cdots$$
$$Y = v \ .$$

We invert these equations to obtain u and v and substitute in (19). Then*

$$F(X,Y) = F_{00} + F_{10}X + F_{20}X^2 + F_{11}XY + F_{02}Y^2 + \cdots \ , \qquad F_{10} \neq 0.$$

Let us suppose first** that $F_{10} > 0$ and $F_{02} > 0$.

We shall now use formula (14) to calculate $h_o(t)$ in the neighborhood of $X = 0$, $Y = 0$. However where we formerly decomposed $F(X,Y)$ into $F_0 + F_1$, we shall find it more convenient in the future to write

$$F(X,Y) = F_{00} + F_0 + F_1$$

and let ξ in (9) be the new F_0. This means merely that we must replace t in (14) by $t - F_{00}$. With this new understanding as to the meaning of ξ, let us choose

(21) $$\xi = F_{10}X + F_{02}Y^2 \ ;$$

then the transformation (9) from X, Y to ξ, η is defined by

$$X = \frac{\xi \cos^2 \eta}{F_{10}} \ , \qquad Y = \frac{\xi^{1/2} \sin \eta}{(F_{02})^{1/2}} \ .$$

We now use (15) and (14) to obtain for $t > F_{00}$

---

* It follows from (20) that $\partial u/\partial Y = 0$ at $u = 0$, $v = 0$. See, for example, Courant, [11].

** We suppose here that $F_{02}(0,0) \neq 0$. This means that the curvature of the contour line through (0,0), the line for which $F(X,Y) = t_o = F_{00}$, does not coincide with the curvature of the boundary at the point (0,0). The case of $F_{02} = 0$ could be treated by the method of the next section.

$$(22)\qquad h_o(t-F_{00}) = \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(t-F_{00})^{1/2}}{F_{10}F_{02}^{1/2}} \cos\eta \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(t-F_{00})^{\lambda+\mu/2}}{F_{10}^{\lambda}\, F_{02}^{\mu/2}}$$

$$\cdot \cos^{2\lambda}\eta\, \sin^{\mu}\eta \sum_{p=0}^{\lambda} \sum_{q=0}^{\mu} G_{\lambda-p,\mu-q}\, F_{r,p,q}\, d\eta \quad .$$

For $t < F_{00}$ there are no contour lines in D_o and therefore $h_o(t-F_{00}) = 0$ (see Fig. 3).

The integral in (22) vanishes unless μ is even. Also $F_{r,p,q}$ is zero if $2p + q < 3r$ in view of the choice of ξ in (21).

Hence

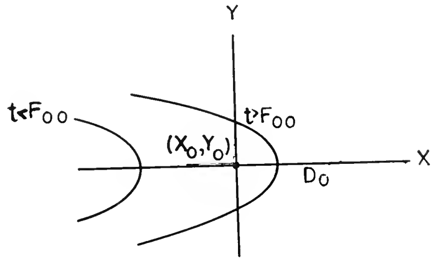

Fig. 3

$$(23)\qquad h_o(t-F_{00}) = \sum_{m=0}^{\infty} \sum_{r=0}^{2m} \frac{(t-F_{00})^{m+1/2}}{(m+1/2)!} \sum_{\lambda=0}^{m+r} \frac{(m+r-\lambda-1/2)!\,\lambda!}{F_{10}^{\lambda+1} F_{02}^{m+r-\lambda+1/2}}$$

$$\cdot \sum_{p=0}^{\lambda} \sum_{q=0}^{2(m+r-\lambda)} G_{\lambda-p,2m+2r-2\lambda-q}\, F_{r,p,q} \; .$$

The application of Erdélyi's theorem now shows that the contribution of this critical value F_{00} of t to the asymptotic expansion of J is

$$(24)\qquad \sum_{m=0}^{\infty} \frac{e^{ikF_{00}+\frac{i\pi}{2}(m+\frac{3}{2})}}{k^{m+3/2}\,|F_{02}|^{1/2}} \sum_{r=0}^{2m} \sum_{\lambda=0}^{m+r} \frac{\lambda!\,(m+r-\lambda-\frac{1}{2})!}{F_{10}^{\lambda+1}\, F_{02}^{m+r-\lambda}} \sum_{p=0}^{\lambda} \sum_{q=0}^{2(m+r-\lambda)} G_{\lambda-p,2m+2r-2\lambda-q} F_{r,p,q} \, .$$

We have put $|F_{02}|^{1/2}$ in (24) because it will be seen shortly that (24) is then applicable to other cases with small modification.

We now consider the case where $F_{10} < 0$ and $F_{02} > 0$ (Fig. 4). The domain D_o lies to the right of $Y = 0$. We can, however, consider a domain D_o' which surrounds the point X_o, Y_o, e.g., the domain ABCDE. In this domain $h_o(t)$ is analytic for all t near t_o because at lease one of f_x and f_y is not zero at (x_o,y_o) and (x_o,y_o) is an interior point of D_o'. Then the $h_o(t)$ which corresponds to the path in D_o itself is the difference of an analytic function and the value of $h_o(t)$ for the path lying in BCD. But the domain BCD occupies the same position with respect to the axis $X = 0$ as the case already treated. Hence the final result is the same as (24).

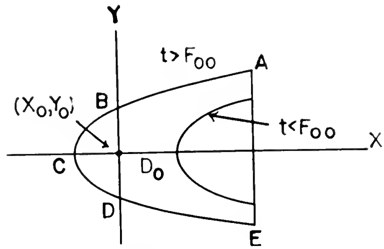

Fig. 4

When $F_{10} > 0$ and $F_{02} < 0$, the part of $h_o(t)$ which contributes to the asymptotic expansion is zero for $t > F_{00}$ but, for $t < F_{00}$, (23) is altered by having $(t-F_{00})^{m+1/2}$ and $F_{02}^{1/2}$ replaced by $-(-1)^m(F_{00}-t)^{m+1/2}$ and $(-F_{02})^{1/2}$ respectively. Thus (24) is unchanged apart from $e^{i\pi(m+3/2)/2}$ being replaced by $e^{i\pi(m+1/2)/2}$. The same result holds for $F_{10} < 0$ and $F_{02} < 0$.

4.2 We consider next the type of critical non-stationary boundary point (x_o,y_o) such that the contour line through (x_o,y_o) coincides with the boundary over a finite length (see Fig. 5). The coordinates X and Y are chosen as in the case of 4.1. However because the contour line and boundary will possess the same curvature at (X_o,Y_o) we have

$$F(X,Y) = F_{00} + F_{10}X + F_{20}X^2 + F_{11}XY + F_{30}X^3 + \dots, \qquad F_{10} \neq 0.$$

We choose ξ and η thus:

$$\xi = F_{10}X, \qquad \eta = Y$$

and consider first the case $F_{10} > 0$. Then $h_o(t) = 0$ for $t < F_{00}$. For $t > 0$, we use (15) and (14) to obtain

$$(25) \qquad h(t-F_{00}) = \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \int_a^b \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(t-F_{00})^{\lambda}}{F_{10}^{\lambda+1}} \eta^{\mu} \sum_{p=0}^{\lambda} \sum_{q=0}^{\mu} G_{\lambda-p,\mu-q} F_{r,p,q} d\eta .$$

Consequently there is a contribution to the asymptotic expansion of J, which because $h(t-F_{00})$ possesses continuous derivatives of all orders, will consist of the zeroth and positive integral powers of $1/k$. However the coefficients of this expansion

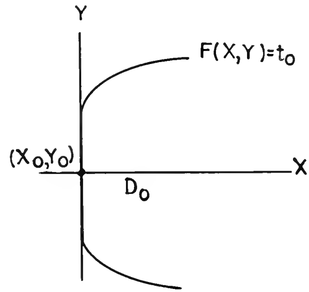

Fig. 5

will depend upon the length of arc over which the boundary of D and the contour line $f(x,y) = t_o$ coincide. They will therefore not be given explicitly.

4.3 We consider next the contribution of a corner in the boundary of D to the asymptotic expansion of J. More precisely state, we consider a point (x_o,y_o) on the boundary of D at which the direction of the tangents to the boundary changes discontinuously but such that the direction of the tangent to the contour line $f(x,y) = t_o$ through (x_o,y_o) does not coincide with either the right or left-hand directions of the tangents to D at (x_o,y_o).

We shall determine a change of variables from (x,y) to (X,Y) so that the boundary of D at (x_o,y_o) will fall along the upper half of the Y-axis and along the right half of the X-axis, so that the domain D_o will consist of the region $X \geq 0$, $Y \geq 0$ near $X_o = 0$, $Y_o = 0$.

Let the boundary arcs on either side of (x_o,y_o) be given respectively by $\phi(x,y) = 0$ and $\Psi(x,y) = 0$. Moreover, since the tangent to D at (x_o,y_o) changes discontinuously, we have at (x_o,y_o)

$$\begin{vmatrix} \phi_x & \phi_y \\ \Psi_x & \Psi_y \end{vmatrix} \neq 0.$$

Since we presuppose that the tangent to the contour line $f(x,y) = t_o$ does not coincide with either of the above tangents, we have also

$$(26) \qquad \begin{vmatrix} \phi_x & \phi_y \\ f_x & f_y \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} \Psi_x & \Psi_y \\ f_x & f_y \end{vmatrix} \neq 0.$$

We consider the linear transformation

$$x - x_o = au + bv \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1,$$
$$y - y_o = cu + dv,$$

and choose a, b, c, d to satisfy the conditions

$$a\,\Psi_x + c\,\Psi_y = 0$$
$$b\,\phi_x + d\,\phi_y = 0,$$

and make the positive u-axis and positive v-axis lie along the tangents to D at (x_o,y_o). In terms of u and v, then, the expansions of $\phi(x,y)$ and $\Psi(x,y)$ are

$$\bar{\phi}(u,v) = \bar{\phi}_{10}u + \bar{\phi}_{20}u^2 + \bar{\phi}_{11}uv + \bar{\phi}_{02}v^2 + \ldots$$
$$\bar{\Psi}(u,v) = \bar{\Psi}_{01}v + \bar{\Psi}_{20}u^2 + \bar{\Psi}_{11}uv + \bar{\Psi}_{02}v^2 + \ldots$$

and

$$(27) \qquad \bar{f}(u,v) = \bar{f}_{00} + \bar{f}_{10}u + \bar{f}_{01}v + \bar{f}_{20}u^2 + \ldots$$

wherein $\bar{f}_{10} \neq 0$ and $\bar{f}_{01} \neq 0$ because of (26).

To make the boundary arcs of D near (x_o,y_o) coincide with the X- and Y-axes we let

$$\bar{\phi}_{10}X = \bar{\phi}_{10}u + \bar{\phi}_{20}u^2 + \ldots$$
$$\bar{\Psi}_{01}Y = \bar{\Psi}_{01}v + \bar{\Psi}_{20}u^2 + \ldots\ .$$

We invert this transformation and substitute in (27). Then

$$F(X,Y) = F_{00} + F_{10}X + F_{01}Y + \dots , \quad F_{10} \neq 0, \quad F_{01} \neq 0.$$

Let

$$\xi = F_{10}X + F_{01}Y.$$

Suppose that $F_{10} > 0$ and $F_{01} > 0$. Then the paths of integration are shown in Fig. 6. Accordingly we make the transformation

$$X = \frac{\xi}{F_{10}} \cos^2 \eta , \quad Y = \frac{\xi}{F_{01}} \sin^2 \eta .$$

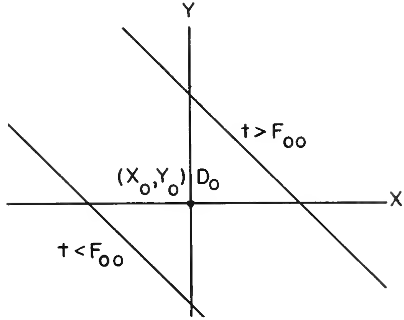

Fig. 6

Now $h_o(t) = 0$ for $t < F_{00}$ because the contour lines $f(x,y) = t$ lie outside of D_o. For $t > F_{00}$ we use equations (17) and (14) to obtain

$$h_o(t-F_{00}) = \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \int_0^{\frac{\pi}{2}} \frac{2(t-F_{00})}{F_{10}\,F_{01}} \sum_{\mu=0}^{\infty} (t-F_{00})^{\mu} \sum_{\lambda=0}^{\mu} \frac{\cos^{2\lambda+1}\eta \, \sin^{2\mu-2\lambda+1}\eta}{F_{10}^{\lambda}\, F_{10}^{\mu-\lambda}}$$

(28)

$$\cdot \sum_{p=0}^{\lambda} \sum_{q=0}^{\mu-\lambda} G_{\lambda-p,\mu-\lambda-q}\, F_{r,p,q}\, d\eta \quad .$$

Since $F_{r,p,q} = 0$ for $p + q < 2r$ we obtain

$$h_o(t-F_{00}) = \sum_{m=0}^{\infty} \sum_{r=0}^{m} \frac{(t-F_{00})^{m+1}}{(m+1)!} \sum_{\lambda=0}^{m+r} \frac{\lambda!\,(m+r-\lambda)!}{F_{10}^{\lambda+1}\, F_{01}^{m+r-\lambda+1}}$$

$$\sum_{p=0}^{\lambda} \sum_{q=0}^{m+r-\lambda} G_{\lambda-p,m+r-\lambda-q}\, F_{r,p,q} \, .$$

Hence the contribution of the point (x_o,y_o) to the asymptotic expansion of (1) is

$$(29) \quad \sum_{m=0}^{\infty} \frac{e^{ikF_{00}+\frac{i\pi}{2}(m+2)}}{k^{m+2}} \sum_{r=0}^{m} \sum_{\lambda=0}^{m+r} \frac{\lambda!\,(m+r-\lambda)!}{F_{10}^{\lambda+1}F_{01}^{m+r-\lambda+1}} \sum_{p=0}^{\lambda} \sum_{q=0}^{m+r-\lambda} G_{\lambda-p,m+r-\lambda-q}F_{r,p,q} \, .$$

When $F_{10} < 0$ and $F_{01} > 0$, the behavior of the contour lines near $X_o = 0$, $Y_o = 0$ is shown in Fig. 7. The domain D_o is still in the first quadrant surrounding (0,0). Let us add to the domain D_o the domain D_1 shown in the figure. Then the boundary of the domain $D_o + D_1$ is the X-axis. This boundary is analytic and the contour lines cut the boundary. Hence $h(t-F_{00})$ taken over $D_o + D_1$ is analytic for all t near $t = F_{00}$. If we now subtract from this $h(t-F_{00})$ the $h_o(t-F_{00})$ taken over the contour lines lying in D_1, we shall obtain the correct form of $h_o(t-F_{00})$ for paths in D_o. The treatment of $h_o(t-F_{00})$ in D_1 is precisely the same as that just given. Hence (28) and (29) obtain for this case too.

When $F_{10} < 0$ and $F_{01} < 0$ the paths $t \gtrless F_{00}$ as given in Fig. 6 are interchanged. Hence (29) gives the result for this case too. Likewise the case $F_{10} > 0$ and $F_{01} < 0$ yields the result (29).

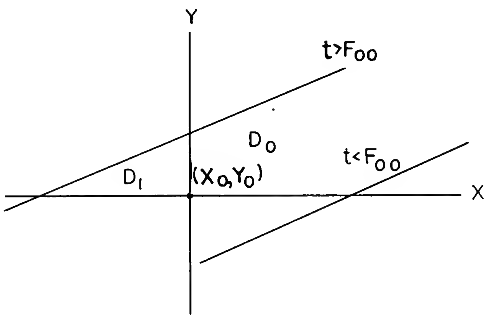

Fig. 7

4.4 We consider as the final case of a critical non-stationary boundary point an (x_o, y_o) where the direction of the tangent to the boundary of D changes discontinuously and one part of the boundary coincides with the contour line $f(x,y) = \text{const.}$ through (x_o, y_o). As in the case treated in (4.2) we shall obtain a series of powers of $\frac{1}{k}$ whose coefficients depend upon the domain of integration and will therefore not be given.

5. Contributions from interior stationary points

We consider next interior points of D at which both f_x and f_y are zero.

5.1 Let the point (x_o,y_o) be a relative minimum or maximum of $f(x,y)$. By a simple rotation of coordinates we may write

$$F(X,Y) = F_{00} + F_{20}(X-X_o)^2 + F_{02}(Y-Y_o)^2 + \dots \quad . \tag{30}$$

In a small neighborhood D_o of (X_o,Y_o) the contour lines F = const. are closed curves surrounding (X_o,Y_o) (see Fig. 8). We shall use (17) and (14) to calculate $h_o(t)$ in D_o. Suppose, firstly, that $F_{20} > 0$ and $F_{02} > 0$. Then let

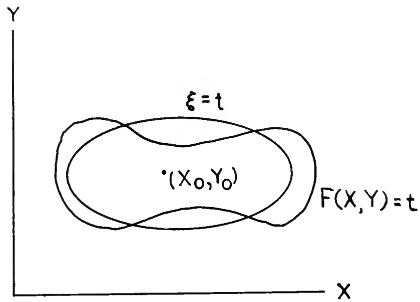

Fig. 8

$$X - X_o = \xi^{1/2} \frac{\cos\eta}{F_{20}^{1/2}} , \qquad Y - Y_o = \xi^{1/2} \frac{\sin\eta}{F_{02}^{1/2}} ,$$

so that

$$\xi = F_{20}(X-X_o)^2 + F_{02}(Y-Y_o)^2 \quad . \tag{31}$$

Now $h(t-F_{00}) = 0$ for $t < F_{00}$. By (17) and (14) we obtain for $t > F_{00}$

$$h_o(t-F_{00}) = \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \int_0^{2\pi} \frac{1}{2(F_{20}F_{02})^{1/2}} \sum_{\mu=0}^{\infty} \sum_{\lambda=0}^{\mu} (t-F_{00})^{\mu/2} \frac{\cos^{\lambda}\eta \, \sin^{\mu-\lambda}\eta}{F_{20}^{\lambda/2} \, F_{02}^{\mu-\lambda/2}} \tag{32}$$

$$\cdot \sum_{p=0}^{\lambda} \sum_{q=0}^{\mu-\lambda} G_{\lambda-p,\mu-\lambda-q} \, F_{r,p,q} \, d\eta \quad .$$

The integral vanishes except when λ and $\mu-\lambda$ are even. Furthermore, in view of the choice of ξ in (31), $F_{r,p,q} = 0$ for $p + q < 3r$. Hence for $t > F_{00}$,

$$h_o(t-F_{00}) = \frac{1}{(F_{20}F_{02})^{1/2}} \sum_{m=0}^{\infty} \sum_{r=0}^{2m} \frac{(t-F_{00})^m}{m!} \sum_{\lambda=0}^{m+r} \frac{(\lambda-\frac{1}{2})!(m+r-\lambda-\frac{1}{2})!}{F_{20}^{\lambda} F_{02}^{m+r-\lambda}} \tag{33}$$
$$\cdot \sum_{p=0}^{2\lambda} \sum_{q=0}^{2r+2m-2\lambda} G_{2\lambda-p,2r+2m-2\lambda-q} F_{r,p,q} .$$

Consequently, by Erdelyi's theorem, the contribution of this critical point to the asymptotic expansion of J is

$$\frac{e^{ikF_{00}}}{|F_{20}F_{02}|^{1/2}} \sum_{m=0}^{\infty} \frac{e^{i\frac{\pi}{2}(m+1)}}{k^{m+1}} \sum_{r=0}^{2m} \sum_{\lambda=0}^{m+r} \frac{(\lambda-\frac{1}{2})!(m+r-\lambda-\frac{1}{2})!}{F_{20}^{\lambda} F_{02}^{m+r-\lambda}} \tag{34}$$
$$\cdot \sum_{p=0}^{2\lambda} \sum_{q=0}^{2r+2m-2\lambda} G_{2\lambda-p,2r+2m-2\lambda-q} F_{r,p,q} .$$

The modulus of $F_{20}F_{02}$ has been introduced to facilitate comparison with other cases.

When $F_{20} < 0$ and $F_{02} < 0$, the result is the same as (33) except that $t > F_{00}$ and $t < F_{00}$ are interchanged. Hence we obtain (34) with the sign reversed.

5.2 Let the point (x_o,y_o) be a saddle point of $f(x,y)$, so that $f_{xx}f_{yy} - f_{xy}^2 < 0$. The coordinates X, Y are chosen so that (30) holds. We assume, firstly, that

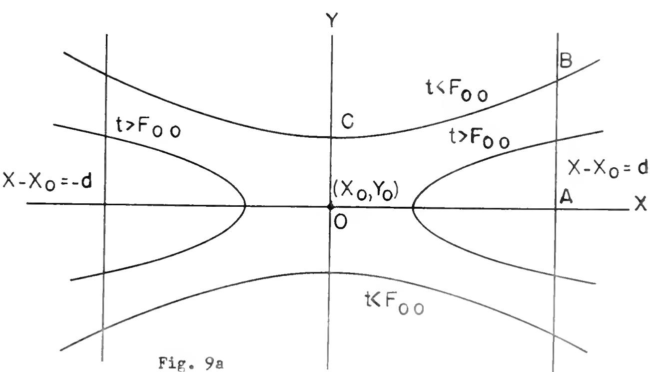

Fig. 9a

$F_{20} > 0$ and $F_{02} < 0$. Then we shall evaluate $h_o(t)$ between $X - X_o = -d$ and $X - X_o = d$ (see Fig. 9a). The path of integration is given by

$$\xi = F_{20}(X-X_o)^2 + F_{02}(Y-Y_o)^2 ,$$

so that the t interval $|t-t_o| < \delta$ is covered by the family of hyperbolas corresponding to $\xi > 0$, $\xi = 0$, and $\xi < 0$. Let $\eta = X - X_o$. We note that for a given t, the hyperbola $\xi = t-F_{00}$ is symmetric with respect to the X and Y axes. Hence we shall see in a moment that we need consider only that part of the path which lies in the first quadrant to evaluate (14).

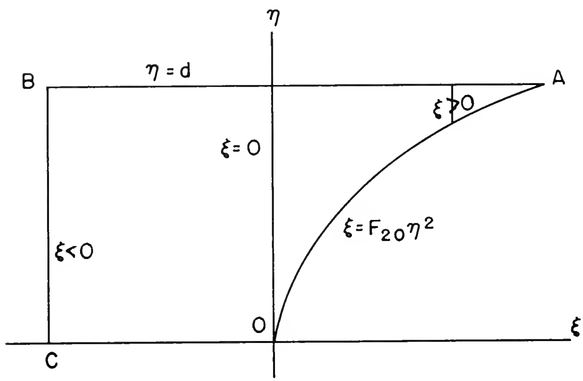

Fig. 9b

Using the notation of article 2, let us consider the integral

$$(35) \qquad \iint_{D_o} \frac{(-1)^r}{r!} G(X,Y)F_1^r(X,Y)\delta(t-F_o)dXdY.$$

If we transform from X, Y to ξ, η in accordance with (9) and write the resulting double integral as a repeated integral with respect to η and then ξ we obtain

$$\int_{\xi_1}^{\xi_2} \delta(t-\xi) \int_{\eta_1(\xi)}^{\eta_2(\xi)} \frac{(-1)^r}{r!} \mathcal{G}(\xi,\eta)\, \mathcal{F}_1^r(\xi,\eta) \frac{\partial(X,Y)}{\partial(\xi,\eta)}\, d\eta\, d\xi.$$

Let $\mathcal{K}(\xi)$ denote the inner integral. Then this double integral becomes

$$\int_{\xi_1}^{\xi_2} \varepsilon(t-\xi)\mathcal{K}(\xi)d\xi$$

or

$$\mathcal{K}(t)$$

which is

$$\int_{\eta_1(t)}^{\eta_2(t)} \frac{(-1)^r}{r!} \mathcal{G}(t,\eta)\, \mathcal{F}_1^r(t,\eta)\, \frac{\partial(X,Y)}{\partial(t,\eta)} \quad d\eta .$$

We see therefore that each term of (14) can be regarded as coming from a term such as (35). If however we evaluate (35) over a path symmetrical with respect to both the X- and Y-axes, such as an hyperbola of Fig. 9a, and if we use the expansion (15) for the first three factors of the integrand, then we need retain only those terms in the expansion which contain even powers of both $X-X_o$ and $Y-Y_o$. Moreover, for these terms the integral (14) is four times the integral taken over the path in the (ξ,η) plane (see Fig. 9b) corresponding to that part of the hyperbola which lies in the first quadrant. Hence we obtain for (14)*

$$h_o(t-F_{00}) = -4 \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \int_{\left(\frac{t-F_{00}}{F_{20}}\right)^{1/2} H(t-F_{00})}^{d} \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} \eta^{2\lambda} \frac{(F_{20}\eta^2-t+F_{00})^{\mu-1/2}}{2F_{02}(-F_{02})^{\mu-1/2}}$$

(36)

$$\cdot \sum_{p=0}^{2\lambda} \sum_{q=0}^{2\mu} G_{2\lambda-p,2\mu-q}\, F_{r,p,q}\, d\eta .$$

We consider therefore integrals of the form

*In the domain of Fig. 9a, $Y - Y_o$ is positive in the first quadrant and hence we write

$$Y - Y_o = \sqrt{\frac{F_{20}\eta^2 - \xi}{-F_{02}}} .$$

$$I_{m,n} = \int_{T^{1/2}H(T)/F_{20}^{1/2}}^{d} \eta^{2m}(F_{20}\eta^2 - T)^{n-1/2}\, d\eta$$

where $T = t - F_{00}$, and $H(T)$ is the Heaviside unit function, i.e., $H(T) = 0$ for $T < 0$ and $H(T) = 1$ for $T > 0$.

By integration,

$$I_{m,n} = \frac{d^{2m+1}(F_{20}d^2 - T)^{n-1/2}}{2m + 2n} - \frac{2n-1}{2m+2n} TI_{m,n-1} \quad \text{for } n \geq 1.$$

At $t = F_{00}$, that is, at $T = 0$, the first term is continuous and has continuous derivatives of all orders. Hence it may be neglected because it will not contribute to the asymptotic expansion of J. Continuing the integration we obtain

$$I_{m,n} = \frac{(-1)^n(2n-1)(2n-3)\ldots 1}{(2m+2n)(2m+2n-2)\ldots(2m+2)} T^n I_{m,o} .$$

Further,

$$I_{m,o} = \frac{d^{2m-1}(F_{20}d^2 - T)^{1/2}}{2mF_{20}} + \frac{2m-1}{2m} \frac{T}{F_{20}} I_{m-1,0} \quad \text{for } m \geq 1.$$

Once again the first term may be neglected insofar as contribution to the asymptotic expansion is concerned, so that

$$I_{m,o} = \frac{(2m-1)(2m-3)\ldots 1}{2m(2m-2)\ldots 2} \frac{T^m}{F_{20}^m} I_{0,0} .$$

Since

$$I_{0,0} = \frac{1}{F_{20}^{1/2}} \log \frac{d + (d^2 - T/F_{20})^{1/2}}{|T|^{1/2} \quad F_{20}^{1/2}}$$

and since the numerator in the logarithmic factor will not contribute to the asymptotic expansion, the significant part of $I_{m,n}$ is given by

$$I_{m,n} = (-1)^{n+1} \frac{(m-1/2)!(n-1/2)!}{(m+n)!2\pi F_{20}^{m+1/2}} T^{m+n} \log |T|.$$

Therefore the relevant part of $h_o(t-F_{00})$ is

$$4 \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} (-1)^{\mu} \frac{(\lambda - \frac{1}{2})! \ (\mu - \frac{1}{2})!}{2F_{02}(-F_{02})^{\mu-1/2}(\lambda+\mu)!2\pi F_{20}^{\lambda+1/2}} (t-F_{00})^{\lambda+\mu}$$

$$\cdot \log|t-F_{00}| \sum_{p=0}^{2\lambda} \sum_{q=0}^{2\mu} G_{2\lambda-p,2\mu-q} F_{r,p,q} .$$

Since we can ignore continuous functions with continuous derivatives we can replace

$$\frac{\partial^r}{\partial t^r} (t-F_{00})^{\lambda+\mu} \log|t-F_{00}|$$

by

$$\frac{(\lambda+\mu)!}{(\lambda+\mu-r)!} (t-F_{00})^{\lambda+\mu-r} \log|t-F_{00}|.$$

Also, $F_{r,p,q} = 0$ if $p + q < 3r$, so that the significant part of $h_o(t-F_{00})$ is

$$\frac{1}{(-F_{20}F_{02})^{1/2}} \sum_{m=0}^{\infty} \sum_{r=0}^{2m} \frac{(t-F_{00})^m}{m!\pi} \log|t-F_{00}| \sum_{\lambda=0}^{m+r} \frac{(\lambda- \frac{1}{2})!(m+r-\lambda- \frac{1}{2})!}{F_{20}^{\lambda} \ F_{02}^{m+r-\lambda}}$$

(37)
$$\cdot \sum_{p=0}^{2\lambda} \sum_{q=0}^{2r+2m-2\lambda} G_{2\lambda-p,2r+2m-2\lambda-q} F_{r,p,q} .$$

Use of Theorem 4 of Appendix A now shows that the contribution to the asymptotic expansion of J is the same as (34) multiplied by i.

This result remains unaltered if $F_{20} < 0$ and $F_{02} > 0$.

5.3 Let the point (x_o, y_o) be a double point, that is $f_{xx}f_{yy} - f_{xy}^2 = 0$. In this case we choose X, Y so that*

$$F(X,Y) = F_{00} + F_{20}(X-X_o)^2 + F_{30}(X-X_o)^3 + F_{21}(X-X_o)^2(Y-Y_o) + F_{12}(X-X_o)(Y-Y_o)^2 + F_{03}(Y-Y_o)^3 + \dots, \qquad F_{20} \neq 0.$$

We assume first that $F_{20} > 0$ and that $F_{03} > 0$, and choose

$$\xi = F_{20}(X-X_o)^2 + F_{03}(Y-Y_o)^3, \quad \eta = X - X_o . \tag{38}$$

We shall evaluate $h_o(t)$ along ξ = const. from $X - X_o = -d$ to $X - X_o = d$. For t-values from $t-F_{00}$ negative to $t-F_{00}$ positive, ξ ranges from negative values to positive ones (see Fig. 10).

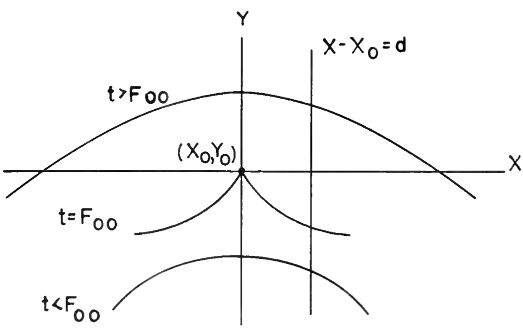

Fig. 10

*If $F_{12} \neq 0$ we can apply a further transformation, $X-X_o = \bar{X} - \bar{X}_o$, $Y-Y_o = \bar{Y} - \bar{Y}_o - \frac{F_{12}}{3F_{03}}(\bar{X} - \bar{X}_o)$, to eliminate the F_{12} term. However this is not necessary in the cases considered here.

However, the argument given in Section 5.2 concerning symmetry of the path of integration can be applied here. In this case we see from the symmetry of the domain of integration with respect to $Y = Y_o$ that we need retain only terms which involve even powers of $X - X_o$ in the expansion (15) and multiply the result by 2. If we then make the change of variables from X, Y to ξ, η and use (14) we obtain

(39) $$h_o(t-F_{00}) = 2\sum_{r=0}^{\infty}\frac{\partial^r}{\partial t^r}\int_0^d \sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\eta^{2\lambda}(T-F_{20}\eta^2)^{(\mu-2)/3}}{3F_{03}\,F_{03}^{(\mu-2)/3}} \cdot \sum_{p=0}^{2\lambda}\sum_{q=0}^{\mu} G_{2\lambda-p,\mu-q}\,F_{r,p,q}\,d\eta ,$$

where, again, $T = t-F_{00}$.

We therefore have to evaluate integrals of the form

$$J_{m,n} = \int_0^d \eta^{2m}\,(T-F_{20}\,\eta^2)^{(n-2)/3}\,d\eta .$$

Retaining only terms which contribute to the asymptotic expansion (cf. section 5.2), we find that

$$J_{m,3s} = \frac{(m-\frac{1}{2})!(\frac{n-2}{3})!(-\frac{1}{6})!T^{m+s}}{(-\frac{1}{2})!(-\frac{2}{3})!(m+\frac{n}{3}-\frac{1}{6})!F_{20}^m}\int_0^d (T-F_{20}\,\eta^2)^{-2/3}\,d\eta ;$$

$$J_{m,3s+1} = \frac{(m-\frac{1}{2})!(\frac{n-2}{3})!(\frac{1}{6})!T^{m+s}}{(-\frac{1}{2})!(-\frac{1}{3})!(m+\frac{n}{3}-\frac{1}{6})!F_{20}^m}\int_0^d (T-F_{20}\,\eta^2)^{-1/3}\,d\eta ;$$

$$J_{m,3s+2} = 0 .$$

Now d is independent of T. Hence

$$\int_0^d (T-F_{20}\,\eta^2)^{-2/3}\,d\eta = \int_0^{\infty}(T-F_{20}\,\eta^2)^{-2/3}\,d\eta$$

aside from a term which is continuous and has continuous derivatives for values of T bounded away from 0. We consider therefore the right-hand integral.

For T > 0

$$\int_0^{\infty} (T-F_{20}\eta^2)^{-2/3} d\eta = \int_0^{(T/F_{20})^{1/2}} (T-F_{20}\eta^2)^{-2/3} d\eta + \int_{(T/F_{20})^{1/2}}^{\infty} (T - F_{20}\eta^2)^{-2/3} d\eta .$$

Make the substitution $\eta = \sqrt{T/F_{20}} \sin\theta$ in the first integral on the right and $\eta = \sqrt{T/F_{20}} \sec\theta$ in the second. Then*

$$\int_0^{\infty} (T-F_{20}\eta^2)^{-2/3} d\eta = \frac{T^{-1/6}}{F_{20}^{1/2}} \int_0^{\frac{\pi}{2}} (\cos^{-1/3}\theta + \sin^{-1/3}\theta \cos^{-2/3}\theta)\, d\theta$$

$$= \frac{(-\frac{2}{3})!\, 3\pi^{1/2}\, T^{-1/6}}{(-\frac{1}{6})!\, 2F_{20}^{1/2}} .$$

For T < 0, we let $\eta = \sqrt{-T/F_{20}} \tan\theta$ to show that

$$\int_0^{\infty} (T-F_{20}\eta^2)^{-2/3} d\eta = \frac{(-T)^{-1/6}}{F_{20}^{1/2}} \int_0^{\frac{\pi}{2}} \cos^{-2/3}\theta\, d\theta = \frac{(-\frac{2}{3})!\, 3^{1/2}\pi^{1/2}}{(-\frac{1}{6})!\, 2F_{20}^{1/2}} (-T)^{-1/6} .$$

The integral for $J_{m,3s+1}$ may be dealt with in a similar way** and we obtain

$$\int_0^{d} (T-F_{20}\eta^2)^{-1/3} d\eta = \frac{(-\frac{1}{3})!\, 3\pi^{1/2} T^{1/6}}{(\frac{1}{6})!\, 2F_{20}^{1/2}} \quad \text{for} \quad T > 0$$

$$= \frac{(-\frac{1}{3})!\, 3^{1/2}\pi^{1/2}(-T)^{1/6}}{(\frac{1}{6})!\, 2F_{20}^{1/2}} \quad \text{for} \quad T < 0,$$

---

* The first term in the integrand on the right may be integrated by a standard integral leading to Gamma functions and the second by a standard integral involving Beta functions. The relationship $\Gamma(x)\Gamma(-x) = \frac{-\pi}{x\sin\pi x}$ may be used to transform intermediate results.

** Before letting the upper limit become infinite we subtract the term $(-F_{20}\eta^2)^{-1/3}$ from the integrand. This term does not involve T and hence does not contribute to the asymptotic expansion. However, it causes the integral with the infinite upper limit to converge.

apart from terms which do not contribute to the asymptotic expansion. Hence

$$J_{m,n} = \frac{(m-\frac{1}{2})!(\frac{n-2}{3})!3^{1/2}|T|^{m+\frac{n}{3}-\frac{1}{6}}}{(m+\frac{n}{3}-\frac{1}{6})!\,2F_{20}^{m+1/2}}\left\{3^{\frac{1}{2}}H(T) + (-1)^{m+[\frac{n}{3}]}H(-T)\right\}, \quad n \neq 3s+2$$

$$= 0 \quad \text{for} \quad n = 3s+2,$$

where $[\frac{n}{3}]$ is the largest integer not greater than $\frac{n}{3}$.

Therefore the significant part of $h_o(t)$ is given by

$$h_o(t-F_{00}) = \frac{1}{3^{1/2}F_{03}} \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(\lambda-\frac{1}{2})!(\frac{\mu-2}{3})!|t-F_{00}|^{\lambda+\frac{\mu}{3}-\frac{1}{6}}}{(\lambda+\frac{\mu}{3}-\frac{1}{6})!F_{20}^{\lambda+1/2}F_{03}^{(\mu-2)/3}}$$

$$\cdot\left\{3^{\frac{1}{2}}H(t-F_{00}) + (-1)^{\lambda+[\frac{\mu}{3}]}H(F_{00}-t)\right\}\sum_{p=0}^{2\lambda}\sum_{q=0}^{\mu} G_{2\lambda-p,\mu-q}F_{r,p,q},$$

where terms in which $\mu = 3s+2$ are absent. Also, the properties of $F_{r,p,q}$ imply that only those terms in which $\lambda + \frac{1}{3}\mu \geq 7r/6$ are present. Hence

(40)
$$h_o(t-F_{00}) = \frac{1}{3^{1/2}} \sum_{r=0}^{\infty} \sum_{m \geq \frac{7r}{2}}^{\infty} \sum_{\lambda=0}^{[\frac{m}{3}]} \frac{(\lambda-\frac{1}{2})!(\frac{m-2}{3}-\lambda)!}{(\frac{m}{3}-r-\frac{1}{6})!F_{20}^{\lambda+1/2}F_{03}^{(m+1)/3-\lambda}} |t-F_{00}|^{\frac{m}{3}-r-\frac{1}{6}}$$

$$\cdot\left\{3^{\frac{1}{2}}H(t-F_{00}) + (-1)^{r+[\frac{m}{3}]}H(F_{00}-t)\right\}\sum_{p=0}^{2\lambda}\sum_{q=0}^{m-3\lambda} G_{2\lambda-p,m-3\lambda-q}F_{r,p,q},$$

where terms in which $m = 3s+2$ are absent.

Consequently, the contribution to the asymptotic expansion of J is, by Erdélyi's theorem,

(41)
$$\frac{e^{ikF_{00}}}{3^{1/2}} \sum_{m=0}^{\infty} \frac{e^{\frac{i\pi}{2}\left\{[\frac{m+2}{3}]+\frac{1}{2}\right\}}}{k^{m/3+5/6}|F_{20}|^{1/2}} \sum_{r=0}^{2m} \sum_{\lambda=0}^{[\frac{m}{3}]+r} \frac{(\lambda-\frac{1}{2})!(\frac{m-2}{3}+r-\lambda)!}{F_{20}^{\lambda}F_{03}^{(m+1)/3+r-\lambda}}$$

$$\cdot\sum_{p=0}^{2\lambda}\sum_{q=0}^{m+3r-3\lambda} G_{2\lambda-p,m+3r-3\lambda-q}F_{r,p,q},$$

where terms in which $m = 3s+2$ are absent.*

When $F_{20} < 0$, $F_{03} > 0$ we must replace $e^{\frac{i\pi}{2}\left\{[\frac{m+2}{3}] + \frac{1}{2}\right\}}$ by $e^{\frac{i\pi}{2}\left\{[\frac{m+2}{3}] - \frac{1}{2}\right\}}$.

When $F_{20} > 0$ and $F_{03} < 0$ we only alter the sign of (41).

When $F_{20} < 0$ and $F_{03} < 0$ we alter the sign of (41) and replace $e^{\frac{i\pi}{2}\left\{[\frac{m+2}{3}] + \frac{1}{2}\right\}}$ by $e^{\frac{i\pi}{2}\left\{[\frac{m+2}{3}] - \frac{1}{2}\right\}}$.

6. Contributions from boundary stationary points

We consider in this article stationary points which lie on the boundary of D. The boundary is assumed to be an analytic curve in the neighborhood of (x_o,y_o).

6.1 We consider first boundary stationary points (x_o,y_o) at which $f(x,y)$ has a maximum or minimum relative to all neighboring interior and boundary points of D. To treat such a point our first step will be to make the boundary of D in the neighborhood of the point the axis of coordinates and, at the same time, to reduce the second-degree terms in the Taylor's expansion of $f(x,y)$ around the point to a sum of squares (see Fig. 11).

Let $\emptyset(x,y) = 0$ be the equation of the boundary in the neighborhood of (x_o,y_o). We first introduce the linear transformation

$$(42) \qquad \begin{aligned} x - x_o &= ah + bk \\ y - y_o &= ch + dk \end{aligned} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

and choose a, b, c, d subject to the condition that $f_{hk}(0,0)$ vanishes. Hence $f(x,y)$ and $\emptyset(x,y)$ become

$$f(h,k) = f_{00} + f_{20}h^2 + f_{02}k^2 + \dots$$

*Focke seems to have omitted the factor i^{τ} in his formula (153). Otherwise his result for this case agrees with our formula (41).

and

$$\phi(h,k) = \phi_{10}h + \phi_{01}k + \phi_{20}h^2 + \phi_{11}hk + \phi_{02}k^2 + \dots \quad .$$

We may also choose a and c so that the sign of $\phi_h(0,0)$ is the sign of $\phi(x,y)$ in D, that is, so that the positive h-axis points into D.

If $\phi_k(0,0)$ is not zero it is possible to make a further transformation*

$$h = \frac{1}{d}(\phi_{10}f_{02}\bar{h} - \phi_{01}\bar{k})$$

$$k = \frac{1}{d}(\phi_{01}f_{20}\bar{h} + \phi_{10}\bar{k}) ,$$

where $d = \phi_{10}^2 f_{02} + \phi_{01}^2 f_{20}$, so as to make $\phi_k = 0$ and leave the form of f(h,k) unaltered. Suppose (x_o, y_o) is a relative minimum; then $f_{20} > 0$ and $f_{02} > 0$, so that $d \neq 0$. As a consequence of this transformation

(43) $$f(\bar{h},\bar{k}) = f_{00} + \bar{f}_{20}\bar{h}^2 + \bar{f}_{02}\bar{k}^2 + \dots$$

and

$$\phi(\bar{h},\bar{k}) = \bar{\phi}_{10}\bar{h} + \bar{\phi}_{20}\bar{h}^2 + \bar{\phi}_{11}\bar{h}\,\bar{k} + \bar{\phi}_{02}\bar{k}^2 + \dots \quad .$$

Now $\phi(\bar{h},\bar{k}) = 0$ is the original boundary curve. Let

(44) $X = \phi(\bar{h},\bar{k}), \quad Y = \bar{k}.$

If we solve the equations (44) for $\bar{h}$ and $\bar{k}$ in terms of X and Y we obtain

$$\bar{h} = a_{10}X + a_{20}X^2 + a_{11}XY + a_{02}Y^2$$

$$\bar{k} = Y.$$

We substitute in (43) and obtain

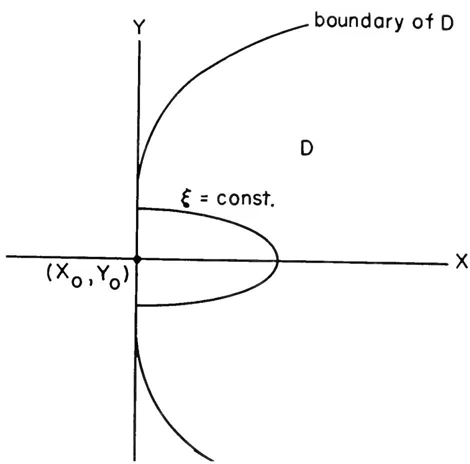

Fig. 11

*This transformation is suggested by Focke [6], p. 46.

(45) $$F(X,Y) = F_{00} + F_{20}X^2 + F_{02}Y^2 + \dots ,$$

where F_{00}, F_{20}, F_{02}, ... are new coefficients and where we have changed the functional symbol to accord with earlier notation.

The above sequence of transformations has transformed the original boundary to $X = 0$ while achieving the form (45) for $F(X,Y)$.

We now proceed as in Section 5.1 except that $X_o = 0$, $Y_o = 0$, and the limits of integration are from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. We obtain for this case

(46) $$h_o(t-F_{00}) = \frac{1}{2(F_{20}F_{02})^{1/2}} \sum_{m=0}^{\infty} \frac{(t-F_{00})^{\frac{m}{2}}}{(m/2)!} \sum_{r=0}^{m} \sum_{\mu=0}^{\left[\frac{m}{2}+r\right]} \frac{(\mu-\frac{1}{2})!(\frac{m}{2}+r-\mu-\frac{1}{2})!}{F_{20}^{m/2+r-\mu}F_{02}^{\mu}} \cdot \sum_{p=0}^{2r-2\mu+2m} \sum_{q=0}^{2\mu} G_{2r+m-2\mu-p,2\mu-q} F_{r,p,q} ,$$

where $[p]$ is the largest integer not greater than p.

Therefore the corresponding asymptotic expansion is

(47) $$\frac{e^{ikF_{00}}}{2(F_{20}F_{02})^{1/2}} \sum_{m=0}^{\infty} \frac{e^{i\frac{\pi}{2}(1+\frac{m}{2})}}{k^{1+m/2}} \sum_{r=0}^{\mu} \sum_{\mu=0}^{\left[\frac{m}{2}+r\right]} \frac{(\mu-\frac{1}{2})!(\frac{m}{2}+r-\mu-\frac{1}{2})!}{F_{20}^{m/2+r-\mu}F_{02}^{\mu}} \cdot \sum_{p=0}^{2r-2\mu+m} \sum_{q=0}^{2\mu} G_{2r+m-2\mu-p,2\mu-q} F_{r,p,q} .$$

If (x_o,y_o) is a relative maximum, then $F_{20} < 0$ and $F_{02} < 0$ and the sign of (47) is also reversed.

6.2 We suppose next that (x_o,y_o) is a saddle point of $f(x,y)$ so that $f_{xx}f_{yy} - f_{xy}^2 < 0$. We introduce the same sequence of transformations used in Section 6.1 to obtain the form (45) for $F(X,Y)$ while making the boundary curve the Y-axis* (see Fig. 12). We suppose first that $F_{20} > 0$ and $F_{02} < 0$.

We now follow the treatment of the interior saddle-point as in Section 5.2 except that the paths of integration lie only in the first and fourth quadrants. Hence we may neglect only those terms in the Taylor's expansion of $\frac{(-1)^r}{r!} G(X,Y)F_1^r(\lambda,Y)$ (compare (35)) which contain odd powers of Y. Instead of (36) we obtain

(48)
$$h(t-F_{00}) = -2 \sum_{r=0}^{\infty} \frac{\partial^r}{\partial t^r} \int_{\frac{(t-F_{00})^{1/2}}{(F_{20})^{1/2}} H(t-F_{00})}^{d} \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} \eta^{\lambda} \frac{(F_{20}\eta^2 - t+F_{00})^{\mu-1/2}}{2F_{02}(-F_{02})^{\mu-1/2}} \cdot \sum_{p=0}^{\lambda} \sum_{q=0}^{2\mu} G_{\lambda-p,2\mu-q} F_{r,p,q} \, d\eta \, .$$

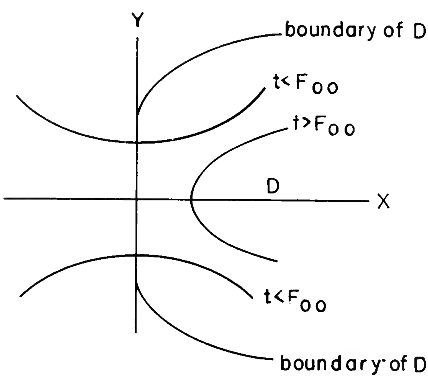

Fig. 12

*In this saddle point case it no longer follows that $d \neq 0$. We assume that $d \neq 0$; this assumption means that the boundary of the domain D does not coincide with either asymptote of the family of hyperboles $f_{20}(x-x_o)^2 + f_{11}(x-x_o)(y-y_o) + f_{02}(y-y_o)^2 = \text{const}$.

Integrals of the type appearing in (48) were already considered in Section 5.2. Evaluation of (48) and the application of Theorem 4 of the Appendix leads to (47) except that $(F_{20}F_{02})^{1/2}$ is replaced by $|F_{20}F_{02}|^{1/2}/i$. This result also holds when $F_{20} < 0$ and $F_{02} > 0$.

7. Remarks on the method

For the benefit of those readers who may have been following some of the general theory of asymptotic solution of Maxwell's equations being developed at New York University we shall relate this theory to that of the present paper. A prime objective of the general theory has been to derive the asymptotic form of time harmonic solutions of Maxwell's equations (or the scalar second order hyperbolic equation) from conditions imposed on the coefficients of the differential equations and the initial and boundary conditions. Of course these conditions would be immediately related to the physical conditions of the problem. To derive this asymptotic series as well as to calculate it this theory uses the concept of the pulse solution of Maxwell's equations. By determining the behavior of the pulse solution a function of x, y, z, and t, in the neighborhood of each of its singularities with respect to t one can write down at once by means of a general theorem the asymptotic series solution of Maxwell's equations[13]. Given the problem of obtaining the asymptotic form of the integrals considered in this paper, one can write down the corresponding pulse solution, proceed to determine its behavior in the neighborhood of its singularities, and then apply the general theorem to obtain the asymptotic form of the integrals. There is, of course, a contribution to this asymptotic value from each singularity of the pulse solution.

The present paper, as has been seen, throws the problem of evaluating the double Fourier integrals asymptotically back to the problem of the single Fourier integral and the utilizes Erdélyi's theorem. The function h(t) of the present paper is precisely the pulse solution of the general theory though it is not obtained in the same way as in the general theory. Moreover, as this paper shows the singularities of $h(t)$ are the critical points of the double integral. Insofar as theory is concerned, the present paper is more general in some respects and less general in others with respect to the kinds of critical points it can handle. However the present paper is decidedly more efficacious in calculating the asymptotic series which corresponds to each critical point. In the present method one proves more readily that the contribution of each critical point to the asymptotic value of the double integral is determined by a small neighborhood of the critical point and by using the form (14) for $h_o(t)$ one obtains more expeditiously the actual asymptotic expansion contributed by each critical point.

Appendix

In this appendix we consider the asymptotic expansion of a Fourier integral in which the integrand has a logarithmic singularity. The analysis runs along lines similar to those followed by Erdélyi [7].

Firstly we discuss

$$I = \int_{\alpha}^{\beta} e^{ikt}(t-\alpha)^{\lambda-1}\, \phi(t)\, \log(t-\alpha)dt,$$

where $0 < \lambda \leq 1$, $\phi(t)$ is N times continuously differentiable for $\alpha \leq t \leq \beta$, and $\phi^{(n)}(\beta) = 0$ for $n = 0, 1, \ldots, N-1$.

Let

$$X_o(t) = e^{ikt}(t-\alpha)^{\lambda-1} \log(t-\alpha)$$

and

$$X_n(t) = \int_{t}^{i\infty} X_{n-1}(u)\, du \quad ,$$

where the path of integration is defined by $u = t + iy$ $(y \geq 0)$. The integral then converges absolutely.

Repeated integration by parts of I gives

$$\text{(A1)} \qquad I = \sum_{n=0}^{N-1} \phi^{(n)}(\alpha)X_{n+1}(\alpha) + \int_{\alpha}^{\beta} X_N(t)\, \phi^{(N)}(t)\, dt.$$

Now

$$X_n(t) = \int_{0}^{\infty} iX_{n-1}(t+iy)\, dy = \frac{e^{\frac{1}{2}in\pi}}{(n-1)!} \int_{0}^{\infty} y^{n-1}X_o(t+iy)dy$$

by repeated interchange of the order of integration. Hence

$$\text{(A2)} \qquad X_n(t) = \frac{e^{\frac{1}{2}in\pi}}{(n-1)!}\, e^{ikt} \int_{0}^{\infty} y^{n-1}e^{-ky}(t+iy-\alpha)^{\lambda-1}\log(t+iy-\alpha)dy.$$

Therefore

$$X_n(\alpha) = \frac{e^{ik\alpha + \frac{1}{2} i\pi(n+\lambda-1)}}{(n-1)!} \int_0^\infty y^{n+\lambda-2} e^{-ky} \left\{ \frac{1}{2} i\pi + \log y \right\} dy.$$

Since

(A3)
$$z! = \int_0^\infty y^z e^{-y} dy,$$
$$z!' = \int_0^\infty y^z e^{-y} \log y \, dy .$$

Thus

$$\int_0^\infty y^z e^{-y} \log y \, dy = z! \, \Psi(z)$$

where

$$\Psi(z) = \frac{z!'}{z!} .$$

The properties of the Ψ-function are well known. For example,

$$\Psi(z+1) = \frac{1}{z+1} + \Psi(z)$$

and

$$\Psi(0) = -\gamma,$$

where γ is Euler's constant.

Employing (A3) we obtain

(A4) $$X_n(\alpha) = \frac{e^{ik\alpha + \frac{1}{2} i\pi(n+\lambda-1)} (n+\lambda-2)!}{(n-1)! \, k^{n+\lambda-1}} \left\{ \frac{1}{2} i\pi - \log k + \Psi(n + \lambda-2) \right\} .$$

Also

$$|t+iy-\alpha|^{\lambda-1} \leq (t-\alpha)^{\lambda-1}$$

and

$$|\log(t+iy-\alpha)| = \left|\log(t-\alpha) + \log\left\{1 + \frac{y^2}{(t-\alpha)^2}\right\}^{1/2} + i\tan^{-1}\frac{y}{t-\alpha}\right|$$

$$\leq \frac{1}{2}\pi + \log(t-\alpha) + \log\left\{1 + \frac{y^{\lambda/2}}{(t-\alpha)^{\lambda/2}}\right\}^{2/\lambda}$$

$$\leq \frac{1}{2}\pi + \log(t-\alpha) + \frac{2y^{\lambda/2}}{\lambda(t-\alpha)^{\lambda/2}} .$$

Hence, from (A2),

$$|X_n(t)| \leq \frac{(t-\alpha)^{\lambda-1}}{k^n}\left\{\frac{1}{2}\pi + \log(t-\alpha)\right\} + \frac{2(t-\alpha)^{(\lambda/2)-1}(n+\frac{1}{2}\lambda-1)!}{\lambda k^{n+\lambda/2}(n-1)!} .$$

Therefore

(A5) $$\int_\alpha^\beta X_N(t)\,\phi^{(N)}(t)dt = O(k^{-N}).$$

Combining (A1), (A4) and (A5) we obtain:

<u>Theorem 1</u>

If $\phi(t)$ is N times continuously differentiable for $\alpha \leq t \leq \beta$, $\phi^{(n)}(\beta) = 0$ for $n = 0, 1, \ldots, N-1$, and $0 < \lambda \leq 1$, then

$$\int_\alpha^\beta e^{ikt}(t-\alpha)^{\lambda-1}\phi(t)\log(t-\alpha)\,dt$$

$$= \sum_{n=0}^{N-1} \frac{e^{ik\alpha + \frac{1}{2}i\pi(n+\lambda)}(n+\lambda-1)!}{n!\,k^{n+\lambda}}\left\{\frac{1}{2}i\pi - \log k + \Psi(n+\lambda-1)\right\}\phi^{(n)}(\alpha) + O(\frac{1}{k^N}).$$

Similarly we can prove:

<u>Theorem 2</u>

If $\phi(t)$ is N times continuously differentiable for $\alpha \leq t \leq \beta$, $\phi^{(n)}(\alpha) = 0$ for $n = 0, 1, \ldots, N-1$, and $0 < \mu \leq 1$, then

$$\int_\alpha^\beta e^{ikt}(\beta-t)^{\mu-1}\phi(t)\log(\beta-t)\,dt$$

$$= \sum_{n=0}^{N-1} \frac{e^{ik\beta+\frac{1}{2}i\pi(n-\mu)}(n+\mu-1)!}{n!k^{n+\mu}} \left\{\Psi(n+\mu-1)-\log k-\frac{1}{2}i\pi\right\} \phi^{(n)}(\beta) + O\left(\frac{1}{k^N}\right).$$

When ϕ does not vanish at either end of the integral the introduction of a neutralizer enables us to deduce the asymptotic expansion from the preceding results. Since the use of the neutralizer has been described by Erdélyi we shall quote the theorem obtained. It is

Theorem 3

If $\phi(t)$ is N times continuously differentiable for $\alpha \le t \le \beta$ and $0 < \lambda \le 1$, $0 < \mu \le 1$, then

$$\int_\alpha^\beta e^{ikt}(t-\alpha)^{\lambda-1}(\beta-t)^{\mu-1}\phi(t)\log(t-\alpha)dt$$

$$= \sum_{n=0}^{N-1} \frac{e^{ik\alpha+\frac{1}{2}i\pi(n+\lambda)}(n+\lambda-1)!}{n!\,k^{n+\lambda}}$$

$$\cdot \left\{\frac{1}{2}i\pi - \log k + \Psi(n+\lambda-1)\right\} \frac{d^n}{d\alpha^n}\left\{(\beta-\alpha)^{\mu-1}\,\phi(\alpha)\right\}$$

$$+ \sum_{n=0}^{N-1} \frac{e^{ik\beta+\frac{1}{2}i\pi(n-\mu)}(n+\mu-1)!}{n!\,k^{n+\mu}}$$

$$\cdot \frac{d^n}{d\beta^n}\left\{(\beta-\alpha)^{\lambda-1}\,\phi(\beta)\,\log(\beta-\alpha)\right\} + O\left(\frac{1}{k^N}\right).$$

Finally, we consider what happens when the logarithmic singularity occurs at an interior point. We restrict attention to

$$\int_\alpha^\beta e^{ikt}\,\phi(t)\,\log(t-c)\,dt,$$

where $\alpha < c < \beta$ and $\log(t-c)$ is defined to be $\log|t-c| + i\pi$ when $t < c$. From Theorem 3,

$$\int_c^\beta e^{ikt}\,\phi(t)\,\log(t-c)\,dt$$

$$= E_\beta + \sum_{n=0}^{N-1} \frac{e^{ikc+\frac{1}{2}i\pi(n+1)}}{k^{n+1}} \left\{ \frac{1}{2}i\pi - \log k + \Upsilon(n) \right\} \phi^{(n)}(c) + O(k^{-N});$$

$$\int_\alpha^c e^{ikt}\,\phi(t)\,\log(t-c)\,dt$$

$$= E_\alpha + \sum_{n=0}^{N-1} \frac{e^{ikc+\frac{1}{2}i\pi(n-1)}}{k^{n+1}} \left\{ \Upsilon(n) - \log k - \frac{1}{2}i\pi \right\} \phi^{(n)}(c)$$

$$+ i\pi \sum_{n=0}^{N-1} \frac{e^{ikc+\frac{1}{2}i\pi(n-1)}}{k^{n+1}} \phi^{(n)}(c) + O(k^{-N}),$$

where E_α, E_β represent contributions from the end-points α and β respectively. Adding these two equations we obtain:

<u>Theorem 4</u>

If $\phi(t)$ is N times continuously differentiable for $\alpha \leq t \leq \beta$, then

$$\int_\alpha^\beta e^{ikt}\phi(t)\,\log(t-c)dt = E_\alpha + E_\beta + O(k^{-N})$$

and

$$\int_\alpha^\beta e^{ikt}\phi(t)\log|t-c|dt = E_\alpha + E_\beta + i\pi \sum_{n=0}^{N-1} \frac{e^{ikc+\frac{1}{2}i\pi(n+1)}}{k^{n+1}} \phi^{(n)}(c) + O(k^{-N}).$$

References

[1] Wolf, E. - The diffraction theory of aberrations; Reports on Progress in Physics, V. XIV, 1951, pp. 95-120, The Physical Society, London.

[2] Bremmer, H. - Diffraction problems of microwave optics; I.R.E. Trans.on antennas and propagation, V.AP-3, 1955, pp. 222-228.

[3] Berghuis, J. - The method of critical regions for two-dimensional integrals and its application to a problem of antenna theory; Ph.D. Thesis, 1955, University of Delft.

[4] Van Kampen, N.G. - An asymptotic treatment of diffraction problems; Physica, V. XIV, 1949, pp. 575-589 and V.XVI, 1950, pp. 817-21.

[5] Bremmer, H. - On the asymptotic evaluation of diffraction integrals with a special view to the theory of defocusing and optical contrast; Physica, V.18, 1952, pp. 469-485.

[6] Focke, J. - Asymptotische Entwicklungen mittels der Methode der stationären Phase; Berichte Über die Verhandlungen der sächsischen Akademie der Wissenschaften zu Leipzig, V. 101, Heft 3, 1954, pp. 1-48.

[7] Erdelyi, A. - Asymptotic expansions; Technical Report 3, O.N.R. 1955. See also Asymptotic representations of Fourier integrals and the method of stationary phase; Jounr. Sci. Indust. Appl. Math., V.3, 1955, pp. 17-27.

[8] Luneberg, R.K. - Mathematical theory of optics; V. II, p. 370, Brown Univ., 1944.

[9] Van der Pol, B. and Bremmer, H. - Operational calculus; Cambridge Univ. Press, Cambridge, 1950, p. 82.

[10] Braun, G. - Zur Methode der stationären Phase; Acta Physica Austriaca, V. 10, 1956, pp. 8-33.

[11] Courant, R. - Differential and integral calculus; Blackie and Son, Ltd., Glasgow, 1937, V.II, p. 152.

[12] Siegel, K.M.,et.al. - Bistatic radar cross sections of surfaces of revolution; Jour. Appl. Phys., V.26, 1955, p. 299.

[13] Kline, M. - Asymptotic solutions of Maxwell's equations involving fractional powers of the frequency; Comm. Pure and Appl. Math., V.8, 1955, pp. 595-614.

[14] Kontorovitch,M.I. and Muravév,U.K. - Derivation of the laws of reflection of geometrical optics on the basis of an asymptotic treatment of the diffraction problem; Jour. of Tech. Physics, Vol. XXII, No. 3, 1952, pp. 394-407.

DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. AF 19(604)1717

(ONE copy unless otherwise noted)

Commander
Air Force Missile Test Center
Patrick Air Force Base, Florida
ATTN: Technical Library

Commander
Air Research and Development Command
Washington 25, D.C.
ATTN: RDDL

Commander
Air Research and Development Command
Washington 25, D.C.
ATTN: Lt. Col. R.F. Bowker, RDTDRC-1

Director
Air University Library
Maxwell Air Force Base, Alabama
ATTN: CR-4582

Headquarters, USAF
Director of Research and Development
Washington 25, D.C.
ATTN: AFDRD

Commander
Wright Air Development Center
Wright-Patterson Air Force Base, Ohio
ATTN: Mr. Walter J. Portune, WCLRS-6

Commander
Wright Air Development Center
Wright-Patterson Air Force Base, Ohio
ATTN: WCLNT-5, N. Draganjac

Commander
Wright Air Development Center
Wright-Patterson Air Force Base, Ohio
ATTN: WCLRE-5, Wave Prop. Section

Air Technical Intelligence Center
Wright-Patterson Air Force Base, Ohio
ATTN: AFOIN-4Bia

Commander
Rome Air Development Center
Griffiss Air Force Base
Rome, New York
ATTN: RCSST-3

Commander
Rome Air Development Center
Griffiss Air Force Base
Rome, New York
ATTN: RCERA-1, D. Mather

Commander
Rome Air Development Center
Griffiss Air Force Base
Rome, New York
ATTN: RCER, Dr. Burgess

Commander
Holloman Air Res. and Dev. Center
Holloman Air Force Base, New Mexico
ATTN: HDOI

Director
Evans Signal Laboratory
Belmar, New Jersey
ATTN: Mr. C.C. Woodyard

Director
Evans Signal Laboratory
Belmar, New Jersey
ATTN: Technical Document Center

Signal Corps Liaison Officer
Mass. Institute of Technology
Cambridge 39, Massachusetts
ATTN: A.D. Bedrosian

Commanding General
SC Engineering Laboratories
Fort Monmouth, New Jersey
ATTN: SCEL Technical Documents Center
Bay 2C122

Department of the Army
Office of the Chief Signal Officer
Washington 25, D.C.
ATTN: SIG RD

Department of the Army
Office of Chief Signal Officer
Engineering and Tech. Division
Washington 25, D.C.
ATTN: SIGNET-5

Director
Ballistics Research Laboratory
Aberdeen Proving Ground, Maryland
ATTN: Ballistics Measurement Lab.

Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland
ATTN: Technical Information Branch
Victor W. Richard

Diamond Ordnance Fuze Laboratories
Washington 25, D.C.
ATTN: Guided Missile Fuze Library
R.D. Hatcher, Chief Microwave
Development Section

(10)Document Service Center
Armed Services Tech. Information Agency
Knott Building
Dayton 2, Ohio
ATTN: DSC-SC

Office of Technical Services
Department of Commerce
Washington 25, D.C.
ATTN: Tech. Reports Section

National Bureau of Standards
Department of Commerce
Washington 25, D.C.
ATTN: Dr. A.G. McNish

National Bureau of Standards
Department of Commerce
Washington 25, D.C.
ATTN: Gustave Shapiro

Boulder Laboratories Library
National Bureau of Standards
Boulder, Colorado
ATTN: Mrs. Victoria S. Barker, Lib.

Director
National Bureau of Standards
Central Radio Prop. Laboratory
Boulder, Colorado
ATTN: Division 14.0

Library
Division of Public Documents
U.S. Government Printing Office
Washington 25, D.C.

(2)Commander
Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Mass.
ATTN: Electronics Document Room
CROOTR-2E

(3)Commander
Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Mass.
ATTN: CRRD- R.E. Hiatt

(10)Commander
L.G. Hanscom Air Field
Bedford, Mass.
CROOTR-2

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D.C.
ATTN: Code Aer-TD-414

Chief, Bureau of Ships
Navy Department
Washington 25, D.C.
ATTN: Mr. E. Johnston, Code 838

Commander
U.S. Naval Air Missile Test Center
Point Mugu, California
ATTN: Code 366; MT-6

U.S. Naval Ordnance Laboratory
White Oak
Silver Spring 19, Maryland
ATTN: The Library

Commander
U.S. Naval Ordnance
Test Station (Code 753)
China Lake, California

Librarian
U.S. Naval Postgraduate School
Monterey, California

Director
U.S. Naval Research Laboratory
Washington 25, D.C.
ATTN: Code 1110

Director, National Security Agency
Washington 25, D.C.

Director
U.S. Naval Research Laboratory
Washington 25, D.C.
ATTN: Code 2021

Director
U.S. Naval Research Laboratory
Washington 25, D.C.
ATTN: Code 5250, J. Bohnert

Commanding Officer and Director
U.S. Navy Underwater Sound Lab.
Fort Trumbull, New London, Conn.
ATTN: Mr. C.M. Dunn

Chief of Naval Research
Navy Department
Washington 25, D.C.
ATTN: Code 427

(2)Commanding Officer and Director
U.S. Navy Electronics Lab. Lib.
San Diego 52, California

Chief, Bureau of Ordnance
Navy Department
Washington 25, D.C.
ATTN: Technical Library, Code Ad3
Mr. H.E. Rex, Code ReS4

Chief, Bureau of Ordnance
Navy Department
Washington 25, D.C.
ATTN: Mr. C.H. Jackson, Code Re9a

Chief, Bureau of Ordnance
Navy Department
Washington 25, D.C.
ATTN: L.E. Wallingford, Code Re4f

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D.C.
ATTN: Code Aer-EL-931

Chief Bureau of Ships
Navy Department
Washington 25, D.C.
ATTN: L.E. Shoemaker, Code 838D

(2)Commander
U.S. Naval Air Development Center
Johnsville, Pennsylvania

Commander
U.S. Naval Air Test Center
Patuxent River, Maryland
ATTN: ET-315, Antenna Section

Naval Ordnance Laboratory
Corona, California
ATTN: Library

U.S. Naval Ordnance Laboratory
Department of the Navy
Corona, California
ATTN: Dr. Robert L. Conger

U.S. Naval Ordnance Plant
Indianapolis 18, Indiana
ATTN: AD-130 (Library)

Airborne Instruments Lab., Inc.
160 Old Country Road
Mineola, New York
ATTN: Librarian

ACF Electronics
800 N. Pitt Street
Alexandria, Pennsylvania
ATTN: Tech. Library

American Machine and Foundry Company
Electronics Division
1085 Commonwealth Avenue
Boston 15, Mass.
ATTN: Technical Library
Mrs. Rita Moravcsik, Librarian

Andrew Alford Consulting Engineers
299 Atlantic Avenue
Boston 10, Massachusetts
ATTN: Dr. A. Alford

Battelle Memorial Institute
505 King Avenue
Columbus 1, Ohio
ATTN: I-205 F -F
Dr. Bernard H. List

Bell Aircraft Corporation
P.O. Box 1
Buffalo 5, New York
ATTN: Technical Library
Mrs. Jasmine H. Mulcahey

Bell Telephone Labs., Inc.
Whippany, New Jersey
ATTN: Technical Information Lib.

Bendix Aviation Corporation
Pacific Division
11600 Sherman Way
North Hollywood, Calif.
ATTN: Technical Library
J.R. Breninger, Engr. Librarian

Bendix Radio
Div. of Bendix Aviation Corporation
East Joppa Road
Towson 4, Maryland
ATTN: Res. and Development Lab. (465)
Chief Engineer

Bjorksten Research Laboratories, Inc.
P.O. Box 265
Madison 1, Wisconsin
Attn: F.B. Korsgard, Treasurer

Boeing Airplane Company
Pilotless Aircraft Division
Seattle 14, Washington
ATTN: Pilotless Aircraft Div. Library
R.R. Barber, Library Supervisor

Boeing Airplane Company
Wichita Division
Wichita, Kansas
ATTN: Kenneth C. Knight, Librarian

Boeing Airplane Company
Seattle Division
Seattle 24, Washington
ATTN: E.T. Allen, Librarian

Chance Vought Aircraft, Inc.
P.O. Box 5907
Dallas, Texas
ATTN: Engineering Library
Mr. H.S. White, Librarian

Convair, A Division of General Dynamics Corp.
Fort Worth 1, Texas
ATTN: K.G. Brown, Div. Res. Librarian

Convair, Division of General Dynamics Corporation
San Diego 12, California
ATTN: Engineering Library

Cornell Aero. Laboratory, Inc.
4455 Genesee Street
Buffalo 21, New York
ATTN: Elma T. Evans, Librarian

Dalmo Victor Company
1414 El Camino Real
San Carlos, California
ATTN: Dorothea Quigley, Tech. Libr.

Dorne and Margolin, Inc.
30 Sylvester Street
Westbury, L.I., New York

Douglas Aircraft Company, Inc.
Long Beach Division
Long Beach 1, California
ATTN: Engineering Library (C-250)

Douglas Aircraft Company, Inc.
827 Lapham Street
El Segundo, California
ATTN: Engineering Library

Douglas Aircraft Company, Inc.
3000 Ocean Park Boulevard
Santa Monica, California
ATTN: Eq. Sec. Reference Files,
Eq. Eng. A250

Douglas Aircraft Corporation
Tulsa, Oklahoma
ATTN: Technical Library

Electronic Defense Laboratory
P.O. Box 205
Mountain View, California
ATTN: Library

Electronics Research, Inc.
2300 N. New York Avenue
Evansville, Indiana
ATTN: B.H. Baldridge,
Engineering Supervisor

Emerson and Cuming, Inc.
869 Washington Street
Canton, Mass.
ATTN: W.R. Cuming

The Emerson Electric Mfg. Company
8100 Florissant Avenue
St. Louis 21, Missouri
ATTN: Mr. E. Breslin,
Emerson Engineering Library

Fairchild Aircraft
Division Fairchild Eng. and Airplane Corp.
Hagerstown, Maryland
ATTN: Library

Farnsworth Electronics Company
Fort Wayne 1, Indiana
ATTN: Technical Library

Federal Telecommunication Labs.
500 Washington Avenue
Nutley 10, New Jersey
ATTN: Technical Library

Gabriel Electronics
Division Gabriel Company
Needham Heights, Mass.
ATTN: Miss Elaine Foss, Librarian

General Electric Company
Ithaca, New York
ATTN: Advanced Electronics Center
Library, John B. Travis

General Electric Company
Electronics Park
Syracuse, New York
ATTN: Mrs. Anne Damon, Documents
Librarian, Documents Lib.
Bldg. 3-143

General Precision Laboratory, Inc.
63 Bedford Road
Pleasantville, New York
ATTN: Mrs. Mary G. Herbst, Librarian

Goodyear Aircraft Corp.
1210 Massillon Road
Akron 15, Ohio
ATTN: Library D/120 Plant A

Grumman Aircraft Engineering Corp.
Bethpage, L.I., New York
ATTN: Engineering Library,
Plant No. 5

Hallicrafters Corporation
440 West 5th Avenue
Chicago, Illinois
ATTN: Library

Hoffman Laboratories, Inc.
Los Angeles, California
ATTN: Library

Hughes Aircraft Company
Microwave Laboratory
Culver City, California
ATTN: Library, Building 12
M.D. Adcock, Head, Antenna Dept.

Hughes Aircraft Company
Culver City, California
ATTN: Research and Development Lib.

Hycon Eastern, Inc.
75 Cambridge Parkway
Cambridge, Mass.
ATTN: Mr. S. Stein, Library

International Business Machines Corp.
Military Products Division
590 Madison Avenue
New York 22, New York
ATTN: Mr. C.F. McElwain

International Business Machines Corp.
Military Products Division
Airborne Computer Laboratories
Vestal, New York
ATTN: Mr. J.J. Walsh, Dept. 528

Jansky and Bailey, Inc.
1339 Wisconsin Avenue, N.W.
Washington, D.C.
ATTN: Tech. Library, Delmer C. Ports

Henry Jasik, Consulting Engineering
61 Second Street
Mineola, New York

Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California
ATTN: I.E. Newlan

Martin Katzin and Co., Consultants
711 14th Street, N.W.
Washington 5, D.C.

Lincoln Laboratory
P.O. Box 73
Lexington 73, Massachusetts
ATTN: Henry Straus, Document Rm., A-229

Lockheed Aircraft Corporation
Missile Systems Division
7701 Woodley Avenue
Van Nuys, California
ATTN: MSD Library -902, Dept. 82-10
E.A. Blasi, Head, Antenna Lab.

Lockheed Aircraft Corporation
California Division
Engineering Library
Dept. 72-25, Plant A-1, Bldg. 63-1
ATTN: N.C. Harnois

The Glenn L. Martin Company
Denver, Colorado
ATTN: Wm. V. Foley, Engineering Lib.

The Glenn L. Martin Company
Baltimore 3, Maryland
ATTN: Engineering Library
Antenna Design Group

Maryland Electronic Mfg. Corp.
5009 Calvert Road
College Park, Maryland
ATTN: H. Warren Cooper

Mathematical Reviews
80 Waterman Street
Providence 6, Rhode Island

The W.L. Maxon Corporation
460 West 34th Street
New York, New York
ATTN: Antenna Laboratory

McDonnell Aircraft Corporation
Box 516
St. Louis 3, Missouri
ATTN: R.D. Detrich
Engineering Library

McMillan Laboratory, Inc.
Ipswich, Massachusetts
ATTN: Security Officer
Document Room

Melpar, Inc.
3000 Arlington Boulevard
Falls Church, Virginia
ATTN: Engineering, Technical Lib.

Microwave Radiation Company
19223 So. Hamilton Street
Gardena, California
ATTN: Mr. Morris J. Ehrlich, Pres.

Northrop Aircraft, Inc.
Engineering Department
1000 E. Broadway
Hawthorne, California
ATTN: E.A. Freitas, Engineering
Library, Plant III

North American Aviation, Inc.
12214 Lakewood Boulevard
Downey, California
ATTN: Technical Library,
Dept. 991-4

North American Aviation, Inc.
Los Angeles International Airport
Los Angeles 45, California
ATTN: Engineering Technical File

Page Communications Engineers
710 14th Street, N.W.
Washington 5, D.C.
ATTN: Library

Philco Corporation Research Division
A and Lippincott Street
Philadelphia 34, Pa.
ATTN: Technical Library

Pickard and Burns, Inc.
240 Highland Avenue
Needham 94, Mass.
ATTN: Dr. J.T. de Bettencourt

Polytechnic Research and Development
Company, Inc.
202 Tillary Street
Brooklyn 1, New York
ATTN: Technical Library

Radiation Engineering Laboratory
Maynard, Mass.
ATTN: Dr. John Ruze

Radiation, Inc.
P.O. Drawer 37
Melbourne, Florida
ATTN: Technical Library
Mr. M.L. Cox, Assistant
Project Engineer

Radio Corp. of America
RCA Laboratories
Rocky Point, New York
ATTN: P.S. Carter, Lab. Library

RCA Laboratories
Princeton, New Jersey
ATTN: Miss Fern Cloak, Librarian
Research Library

RCA Defense Electronic Products
Camden 2, New Jersey
ATTN: Miss J.M. Steever, Librarian
Library 10-2

(2)The Ramo-Wooldridge Corporation
5730 Arbor Vitae Street
Los Angeles 45, California
ATTN: Margaret C. Whitnah,
Chief Librarian, Research
and Development Library

The Rand Corporation
Via: SRAMA Liaison Office
1700 Main Street
Santa Monica, California
ATTN: Library

Rantic Corporation
Carlabasos, California
ATTN: Technical Library

Raytheon Manufacturing Company
M and R Division
Bedford, Massachusetts
ATTN: Irving Goldstein

Raytheon Laboratory
Wayland, Massachusetts
ATTN: Document Lib., John E. Walsh
Dept. 8974, Antenna

Raytheon Manufacturing Company
Wayland Laboratory
Wayland, Massachusetts
ATTN: Miss Alice G. Anderson, Lib.
Wayland Library

Republic Aviation Corporation
Farmingdale, L.I., New York
ATTN: H. Stieglitz, Engineering Lib.

Rheem Manufacturing Company
Research and Development
9236 East Hall Road
Downey, California
ATTN: J.C. Joerger

Ryan Aeronautical Company
San Diego, California
ATTN: Technical Library

Sanders Associates
137 Canal Street
Nashua, New Hampshire
ATTN: N.R. Wild, Library

Sandia Corporation
P.O. Box 5800
Classified Document Division
Albuquerque, New Mexico

Sperry Gyroscope Company
Great Neck, L.I., New York
ATTN: Engineering Library
Florence W. Turnbull,
Engr. Librarian

Stanford Research Institute
Menlo Park, California
ATTN: Engineering Document Library
Mrs. Beatrice Gibson, Libr.

Sylvania Electric Company
100 First Street
Waltham, Mass.
ATT: Librarian

Systems Laboratories Corporation
15016 Ventura Blvd.
Sherman Oaks, California
ATTN: Dr. W.C. Hoffman

Technical Research Group
17 Union Square West
New York 3, New York

A.S. Thomas, Inc.
161 Devonshire Street
Boston, Mass.
ATTN: A.S. Thomas

Westinghouse Electric Corp.
2519 Wilkens Avenue
Baltimore 3, Maryland
ATTN: Engineering Library
Kent M. Mack

Wheeler Laboratories, Inc.
122 Cutter Mill Road
Great Neck, New York
ATTN: Mr. Harold A. Wheeler

Zenith Plastics Company
Box 91
Gardena, California
ATTN: Mr. S.S. Oleesky

Library
Geophysical Institute
University of Alaska
College, Alaska

Electronic Research Laboratory
Div. of Electronic Engineering
University of California
Berkeley 4, California
ATTN: S. Silver

Electronics Research Laboratory
232 Cory Hall
University of California
Berkeley 4, California
ATTN: J.R. Whinnery

California Institute of Technology
1201 E. California Street
Pasadena, California
ATTN: Dr. Charles H. Papas

Carnegie Institute of Technology
Pittsburgh 13, Pennsylvania
ATTN: Albert E. Heins

School of Electrical Engineering
Cornell University
Ithaca, New York
ATTN: Dr. Henry G. Booker

College of Engineering
University of Florida
Gainesville, Florida
ATTN: Engineering Sciences Library
Professor M.H. Latour

Engineering Experiment Station
Georgia Institute of Technology
Atlanta, Georgia
ATTN: Georgia Tech. Library
Mrs. J.H. Crosland

Technical Reports Collection
Harvard University
Gordon McKay Library
303A Pierce Hall
Oxford Street
Cambridge 38, Massachusetts
ATTN: Mrs. E.L. Hufschmidt, Librarian

Harvard College Observatory
60 Garden Street
Cambridge, Massachusetts
ATTN: Dr. Fred L. Whipple

Serials Department -220-S Library
University of Illinois
Urbana, Illinois
ATTN: Engineering Library

Antenna Section
Electrical Engr. Research Laboratory
University of Illinois
Urbana, Illinois
ATTN: EERL Library, Room 210,
Dr. R.H. DuHamel

The Johns Hopkins University
Department of Physics
Baltimore 18, Maryland
ATTN: Professor Donald E. Kerr

Radiation Laboratory
Johns Hopkins University
1315 St. Paul Street
Baltimore 2, Maryland

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Document Library

Massachusetts Institute of Technology
Research Laboratory of Electronics
Document Room 20B-221
Cambridge 39, Massachusetts

University of Michigan
Electronic Defense Group
Engineering Research Institute
Ann Arbor, Michigan
ATTN: J.A. Boyd, Supervisor

Willow Run Labs., of the Engineering
Research Institute
The University of Michigan
Willow Run Airport
Ypsilanti, Michigan
ATTN: K.M. Siegel, Head, Theory
and Analysis Department

University of Michigan
Willow Run Laboratories
Willow Run Airport
Ypsilanti, Michigan
ATTN: Shelia Coon, Librarian

University of Minnesota
Minneapolis, Minnesota
Engineering Library
ATTN: Head Librarian

Microwave Laboratory
Electrical Engineering Dept.
Northwestern University
Evanston, Illinois
ATTN: Prof. H.E. Beam

Antenna Laboratory
Department of Electrical Engineering
Ohio State University
Columbus, Ohio
ATTN: Dr. T.E. Tice

Oklahoma University Research Found.
Norman, Oklahoma
ATTN: Technical Library
Professor C.L. Farrar

Microwave Research Institute
55 Johnson Street
Brooklyn 1, New York
ATTN: Dr. A.A. Oliner

Polytechnic Institute of Brooklyn
55 Johnson Street
Brooklyn 1, New York
ATTN: Microwave Research Institute

Syracuse University
Dept. of Electrical Engr.
Syracuse 10, New York
ATTN: Technical Library

University of Texas
Electrical Engr. Research Lab.
P.O. Box 8026, Univ. Station
Austin 12, Texas
ATTN: John R. Gerhardt

Dept. of Physics
University of Texas
Austin 12, Texas
ATTN: Physics Library
Claude W. Horton

Tufts University
Medford 55, Massachusetts
ATTN: Research Lab. of Physical Electronics
Professor Charles R. Mingins

University of Toronto
Dept. of Electrical Engineering
Toronto, Ontario, Canada
ATTN: Prof. George Sinclair

University of Washington
Dept. of Electrical Engineering
Seattle, Washington
ATTN: Prof. G. Held

Ionosphere Research Laboratory
Pennsylvania State College
State College, Pennsylvania
ATTN: Prof. A.H. Waynick, Director

Institute of Mathematical Science
25 Waverly Place
New York 3, New York
ATTN: Mrs. JoAn Segal, Librarian

School of Electrical Engineering
Purdue University
Lafayette, Indiana
ATTN: Professor F.V. Schultz

Department of Electrical Engineering
California Institute of Technology
ATTN: Dr. Charles H. Papas

Electronics Division
Rand Corporation
1700 Main Street
Santa Monica, California
ATTN: Dr. Robert Kalaba

National Bureau of Standards
Washington, D.C.
ATTN: Dr. W.K. Saunders

Applied Mathematics and Statistics Lab.
Stanford University
Stanford, California
ATTN: Dr. Albert H. Bowker

Department of Physics and Astronomy
Michigan State University
East Lansing, Michigan
ATTN: Dr. A. Leitner

University of Tennessee
Knoxville, Tennessee
ATTN: Dr. Fred A. Ficken

California Institute of Technology
1201 E. California Street
Pasadena, California
ATTN: Dr. A. Erdelyi

Wayne University
Detroit, Michigan
ATTN: Professor A.F. Stevenson

Mathematics Department
Stanford University
Stanford, California
ATTN: Dr. Harold Levine

University of Minnesota
Minneapolis 14, Minnesota
ATTN: Prof. Paul C. Rosenbloom

Department of Mathematics
University of Pennsylvania
Philadelphia 4, Pa.
ATTN: Professor Bernard Epstein

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Dr. B.S. Gouray

(2) Exchange and Gift Division
The Library of Congress
Washington 25, D.C.

Electrical Engineering Department
Massachusetts Institute of Tech.
Cambridge 39, Massachusetts
ATTN: Dr. L.J. Chu

Nuclear Development Assoc. Inc.
5 New Street
White Plains, New York
ATTN: Library

Lebanon Valley College
Annville, Pennsylvania
ATTN: Professor B.H. Bissinger

Dept. of Physics, Thaw Hall
University of Pittsburgh
Pittsburgh 13, Pa.
ATTN: Dr. Edward Gerjuoy

Dept. of Physics
Amherst College
Amherst, Massachusetts
ATTN: Dr. Arnold Arons

California Institute of Tech.
Electrical Engineering
Pasadena, California
ATTN: Dr. Zohrab A. Kaprielian

Dr. Rodman Doll
209A Emmet St.
Ypsilanti, Michigan

California Institute of Tech.
Pasadena 4, California
ATTN: Mr. Calvin Wilcox

(3) Mr. Robert Brockhurst
Woods Hole Oceanographic Inst.
Woods Hole, Massachusetts

National Bureau of Standards
Boulder, Colorado
ATTN: Dr. R. Gallet

Engineering Library
Plant 5
Grumman Aircraft Corp.
Bethpage, L.I., New York
ATTN: Mrs. A.M. Gray

Mrs. Jane Scanlon
284 South Street
Southbridge, Massachusetts

Dr. Solomon L. Schwebel
3689 Louis Road
Palo Alto, California

Microwave Laboratory
55 Johnson Street
Brooklyn, New York
ATTN: Dr. Bernard Lippmann

University of Minnesota
The University Library
Minneapolis 14, Minnesota
ATTN: Exchange Division

Professor Bernard Friedman
55 Hilltop Avenue
New Rochelle, NewYork

Lincoln Laboratory
Massachusetts Institute of Tech.
P.O. Box 73
Lexington 73, Mass.
ATTN: Dr. Shou Chin Wang, Rm. C-351

Melpar, Inc.
3000 Arlington Boulevard
Falls Church, Virginia
ATTN: Mr. K.S. Kelleher, Section Head

Antenna Laboratory, ERD
HQ Air Force Cambridge Research Center
Laurence C. Hanscom Field
Bedford, Massachusetts
ATTN: Nelson A. Logan

Electronics Research Directorate
HQ Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Massachusetts
ATTN: Dr. Philip Newman, CRRK

HQ Air Force Cambridge Research Center
Laurence C. Hanscom Field
Bedford, Massachusetts
ATTN: Mr. Francis J. Zucker

Crosley AVCO Research Labs.
257 Crescent Street
Waltham, Massachusetts
ATTN: Mr. N.C. Gerson

Antenna Research Section
Microwave Laboratory
Hughes Aircraft Company
Culver City, California
ATTN: Dr. Richard B. Barrar

Columbia University
Hudson Laboratories
P.O. Box 239
145 Palisade Street
Dobbs Ferry, New York
ATTN: Dr. M.W. Johnson

Institute of Fluid Dynamics and Applied Math.
University of Maryland
College Park, Maryland
ATTN: Dr. Elliott Montroll

Dept. of Electrical Engineering
Washington University
Saint Louis 5, Missouri
ATTN: Professor J. Van Bladel

Dept. of the Navy
Office of Naval Research Branch Office
1030 E. Green Street
Pasadena 1, California

Brandeis University
Waltham, Massachusetts
ATTN: Library

General Electric Company
Microwave Laboratory
Electronics Division
Stanford Industrial Park
Palo Alto, California
ATTN: Library

Hughes Research Laboratories
Hughes Aircraft Company
Culver City, California
ATTN: Dr. W.A. Dolid
Bldg. 12, Rm. 2529

Smyth Research Associates
3930 4th Avenue
San Diego 3, California
ATTN: Dr. John B. Smyth

Electrical Engineering
California Institute of Tech.
Pasadena, California
ATTN: Dr. Georges G. Weill

Naval Research Laboratory
Washington 25, D.C.
ATTN: Dr. Henry J. Passerini
Code 5278 A

Dr. George Kear
10585 N. Stelling Rd.
Cupertino, California

Brooklyn Polytechnic Institute
85 Livingston Street
Brooklyn, New York
ATTN: Dr. Nathan Marcuvitz

Department of Electrical Engr.
Brooklyn Polytechnic Institute
85 Livingston Street
Brooklyn, New York
ATTN: Dr. Jerry Shmoys

Dept. of Mathematics
University of New Mexico
Albuquerque, New Mexico
ATTN: Dr. I. Kolodner

W.L. Maxon
460 W. 34th Street
New York, N.Y.
ATTN: Dr. Harry Hochstadt

Date Due

FEB 18 '57	MAR 4 '70		
SEP 1 '61	MAY 23 1969		
SEP 30 '61	AUG 1 9 1970		
OCT 30	NOV 2 [illegible]		
JAN 2 '62	JAN 3 197[illegible]		
[illegible]	FEB 1 [illegible]		
FEB 2[illegible] '62	[illegible] 1976		
APR 2[illegible] '62	MAR [illegible]		
JUL [illegible] '64			
DEC [illegible]			
[illegible]			
MAY 1[illegible] '66	MAY 5 1976		
JUN 8 '66			
JUL 1[illegible] '66	[illegible] 3 1977		
FEB 2[illegible] '67			
DEC [illegible] '68			

NYU
EM
1[illegible]

Jones

Asymptotic expansions of
multiple integrals